Sequences and series - Practice test 4

10 questions | 55 marks | 70 minutes

Applications, counterexamples and model comparisons.

Equivalent correct methods earn the corresponding marks.

1. A display is built in rows containing $4, 7, 10, \ldots$ boxes. [5 marks]

(a) Find the boxes in row 20. [2 marks]

$4 + 19(3) = 61.$

Answer: $61.$

(b) How many boxes are needed for the first 20 rows? [3 marks]

$\frac{20(4+61)}{2} = 650.$

Answer: $650.$

2. A technician measures successive pulses $12,\ -6, 3,\ -1.5, \ldots$ in arbitrary signed units. [6 marks]

(a) Find the eight-pulse total. [3 marks]

$S_8 = \frac{12\left[1-\left(-\frac{1}{2}\right)^8\right]}{\frac{3}{2}} = \frac{255}{32}.$

Answer: $\frac{255}{32} = 7.96875.$

(b) Find the limiting total and tail after eight pulses. [3 marks]

$\frac{12}{1+\frac{1}{2}} = 8;\ 8 - \frac{255}{32} = \frac{1}{32}.$

Answer: Limit 8; tail $\frac{1}{32}$.

3. After a treatment, a stock retains 70% and then gains 12 units. It starts at 10 units. [6 marks]

(a) Find a recurrence and its first two updated values. [2 marks]

Apply the retention before the addition.

Answer: $u_{n+1} = 0.7u_n + 12;\ u_1 = 19, u_2 = 25.3.$

(b) Find an explicit rule and the long-term stock. [4 marks]

$L = \frac{12}{1-0.7} = 40$ and initial deviation -30.

Answer: $u_n = 40 - 30(0.7)^n$; limiting stock 40.

4. AUD 250 is deposited three times, once per year, at 4% annual interest. [6 marks]

(a) Find the value after three years if each deposit is made at the end of a year. [3 marks]

$250[(1.04)^2 + 1.04 + 1] = 780.4.$

Answer: $780.40.$

(b) Find the value if deposits are made at the start of each year, and explain the difference. [3 marks]

The exact model result is $780.4(1.04) = 811.616.$

Answer: 811.62 to nearest cent; each payment earns one extra year of interest.

5. A AUD 20,000 machine can be modelled by losing AUD 2000 yearly or retaining 90% yearly. [6 marks]

(a) Find both values after three years. [2 marks]

$20000-6000=14000;\ 20000(0.9)^3=14580.$

Answer: 14000 and $14580.$

(b) Which model predicts the higher value after three years, and by how much? [2 marks]

$14580-14000=580.$

Answer: The percentage model, by $580.$

(c) Explain the different mathematical mechanisms. [2 marks]

A changing dollar loss under percentage depreciation is not an arithmetic sequence.

Answer: The first has constant differences; the second has a constant ratio.

6. A geometric sequence has first term 2 and ratio 3. Let S_n be its first n-term sum. [7 marks]

(a) Find an explicit expression for S_n. [2 marks]

$\frac{2(3^n-1)}{3-1}=3^n-1.$

Answer: $S_n=3^n-1.$

(b) Prove the result by induction. [5 marks]

Base: $2=3-1$. Assume $S_k=3^k-1$. Add $2(3^k)$ to obtain $3^{k+1}-1$; conclude.

Answer: True for all integers $n\geq 1.$

7. The recurrence $u_{n+1}=u_n+4$ is given without an initial value. [4 marks]

(a) Choose the initial value that makes $u_5=30.$ [2 marks]

$u_5=u_1+4(4)$, so $u_1=14.$

Answer: $u_1=14.$

(b) Can the same recurrence also produce $u_5=10$? Explain. [2 marks]

The recurrence does not fix the starting term.

Answer: Yes; choose $u_1=-6.$

8. A culture has 500 organisms initially and grows by 25% per day in a model. [5 marks]

(a) Give the term rule at whole-number day $n \geq 0$. [2 marks]

The daily growth factor is 1.25.

Answer: $N_n = 500(1.25)^n$.

(b) Find the first whole day with more than 2000 organisms. [3 marks]

$1.25^6 = 3.814697\ldots < 4$ and $1.25^7 = 4.768371\ldots > 4$.

Answer: Day 7.

9. Consider the proposed proof: “Assume $S_{k+1} = (k+1)^2$. Therefore the sum of the first $k+1$ odd numbers is $(k+1)^2$.” [5 marks]

(a) Explain why this is not an induction step. [2 marks]

No implication from the k case is established.

Answer: It assumes precisely the next case that must be proved.

(b) Replace it with a valid step. [3 marks]

Only the previous case is assumed.

Answer: Assume $S_k = k^2$; then $S_{k+1} = S_k + 2k + 1 = (k+1)^2$.

10. Two sequences start $1, 2, 3$. One continues with $u_n = n$; another uses $v_n = n + (n-1)(n-2)(n-3)$. [5 marks]

(a) Check that both rules match the first three terms. [2 marks]

Hence $v_n = n$ for those three indices.

Answer: At $n = 1, 2, 3$, the product term is zero.

(b) Find the fourth terms and explain what this shows about guessing patterns. [3 marks]

$v_4 = 4 + 3(2)(1) = 10$.

Answer: $u_4 = 4, v_4 = 10$; a finite prefix does not uniquely determine a rule.