Matrices and transformations - Practice test 5

10 questions | 63 marks | 70 minutes

Mixed applications: Connect resources, sensor readings, calibration, affine geometry and matrix equations.

Equivalent correct methods earn the corresponding marks.

1. A materials table is $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$, with rows for materials and columns for two products.

[6 marks]

(a) Find the resources needed for 2 of product 1 and 3 of product 2. [3 marks]

Multiply A by $(2, 3)$: $(6 + 3, 4 + 12)$.

Answer: $(9, 16)^T$.

(b) The two material prices are 5 dollars and 2 dollars per unit. Calculate the total resource cost. [3 marks]

Multiply $(5, 2)$ by the resource column.

The total is $45 + 32 = 77$.

Answer: 77 dollars.

2. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 2 \end{pmatrix}$. [6 marks]

(a) Find AB. [4 marks]

Dot the two rows with the two columns.

Answer: $\begin{pmatrix} 0 & 5 \\ 4 & -2 \end{pmatrix}$.

(b) State the order of BA and calculate its middle row. [2 marks]

The middle row of B is $(3, 0)$, so its product with A is three times the first row of A.

Answer: BA has order 3×3; middle row $(3, 0, 6)$.

3. Let $M=\begin{pmatrix} t & 1 \\ 2 & 1 \end{pmatrix}$ and consider $M(x, y)^T = (5, 4)^T$. [6 marks]

(a) For t not equal to 2, solve for x and y in terms of t. [4 marks]

Subtract the two scalar equations to get $(t-2)x = 1$.

Use $y = 4 - 2x$ and simplify.

Answer: $x = \frac{1}{t-2}$, $y = \frac{4t-10}{t-2}$.

(b) Explain what happens at $t = 2$. [2 marks]

The equations then have the same left-hand side but different constants 5 and 4.

Answer: There is no solution.

4. First rotate 90° counterclockwise, then double only the x-coordinate. [6 marks]

(a) Find the combined matrix and describe its area effect. [3 marks]

Multiply the horizontal-stretch matrix by the rotation matrix.

The determinant is 2.

Answer: $T=\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$; area factor 2, orientation preserved.

(b) Find the images of the triangle $(0, 0), (1, 0), (0, 2)$ and its image area. [3 marks]

The rule is (x, y) to $(-2y, x)$.

Its image has perpendicular base 4 and height 1.

Answer: $(0, 0), (0, 1), (-4, 0)$; area 2.

5. Solve $AXB = C$ where $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 6 \\ 3 & 7 \end{pmatrix}$. [7 marks]

(a) Write a correct inverse expression for X. [2 marks]

Undo the left factor on the left and the right factor on the right.

Answer: $X = A^{-1}CB^{-1}$.

(b) Calculate X and verify the product AXB. [5 marks]

First halve the top row of C.

Then multiply on the right by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Multiplying AXB recovers the supplied C.

Answer: $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

6. A transformation is $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ followed by projection onto the x-axis. [6 marks]

(a) Find the combined matrix and explain its singularity. [3 marks]

Projection keeps the first row of A and replaces the second with zero.

The determinant is zero.

Answer: $P_x A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$.

(b) Describe every input giving image $(5, 0)$. [3 marks]

Require $x + 2y = 5$.

The second image equation imposes no extra condition.

Answer: $(5 - 2s, s)$ for real s.

7. A point is rotated 180° about $c = (2, -1)$. [6 marks]

(a) Write the affine coordinate rule and find its fixed point. [3 marks]

Use $q = c - (p - c) = 2c - p$.

Solve $x = 4 - x$ and $y = -2 - y$.

Answer: $(x, y) \mapsto (4 - x, -2 - y)$; fixed point $(2, -1)$.

(b) Can a 2×2 matrix alone describe this rotation in the original coordinates? Explain. [3 marks]

It sends the origin to $(4, -2)$, not the origin.

A translation is required because the centre is not the origin.

Answer: No.

8. A map has matrix $M = \begin{pmatrix} k & 2 \\ 1 & k \end{pmatrix}$. An area-6 region has image area 12. [6 marks]

(a) Find every real value of k. [4 marks]

$|k^2 - 2| = 2$.

The two branches give k^2 equal to 4 or 0.

Answer: $k = 0, 2, -2$.

(b) Classify orientation for every possible k. [2 marks]

At zero the determinant is -2; at either other value it is 2.

Answer: $k = 0$ reverses orientation; $k = 2$ and $k = -2$ preserve it.

9. Two sensors report $u = x + y$ and $v = x - y$. Their outputs are $u = 10$ and $v = 2$. [6 marks]

(a) Use a matrix model to recover x and y. [3 marks]

The coefficient matrix is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ with determinant -2.

Add and subtract the equations, or apply its inverse.

Answer: $x = 6,\ y = 4$.

(b) The second sensor fails, leaving only $u = 10$. Describe all possible inputs and explain the loss of uniqueness. [3 marks]

A single sum does not distinguish these pairs.

The missing second independent equation prevents unique recovery.

Answer: $(t, 10 - t)$ for real t.

10. A calibration sends $(1, 1)$ to $(4, 0)$ and $(1, -1)$ to $(0, 2)$. The map is linear. [8 marks]

(a) Find the calibration matrix. [4 marks]

Half the sum of the image vectors gives $Ae_1 = (2, 1)$.

Half their difference gives $Ae_2 = (2, -1)$.

Answer: $A = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$.

(b) Find the input with calibrated image $(6, 1)$, and state the calibration area factor. [4 marks]

Solve $2x + 2y = 6$ and $x - y = 1$.

The determinant is -4; take its absolute value.

Answer: Input $(2, 1)$; area factor 4.