Matrices and transformations - Practice test 4

10 questions | 63 marks | 70 minutes

Reasoning and proof: Explain matrix claims with proofs, counterexamples and careful dimension checks.

Equivalent correct methods earn the corresponding marks.

1. Let $J=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$. [6 marks]

(a) Apply J to $(1,0)$ and explain why J is not the identity. [3 marks]

The identity must leave $(1,0)$ unchanged.

Answer: Image $(1,1)$.

(b) Find every vector left fixed by J. [3 marks]

Equality $(x+y,x+y)=(x,y)$ implies $y=0$ from the first component and $x=0$ from the second.

Answer: Only the zero vector.

2. Let $A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $B=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. [6 marks]

(a) Calculate AB and explain what this disproves. [3 marks]

Neither factor is zero.

Thus a zero matrix product need not have a zero factor.

Answer: $AB=0$.

(b) Prove that if C is invertible and $CD=0$, then $D=0$. [3 marks]

Multiply on the left by C^{-1}.

Use the inverse identity.

Answer: $D=C^{-1}CD=C^{-1}0=0$.

3. Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $F=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. [7 marks]

(a) Find the conditions on the entries for $MF = FM$. [4 marks]

MF has entries $\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$.

FM has entries $\begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$; equality forces b and c to vanish.

Answer: $b = 0$ and $c = 0$.

(b) If M also commutes with $R=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, what further condition is needed? [3 marks]

For diagonal M, compare the off-diagonal entries of MR and RM.

Answer: $a = d$, so M is a scalar identity.

4. The matrix $P=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projects onto the x-axis. [6 marks]

(a) Show that applying P twice has the same effect as once. [3 marks]

Direct multiplication gives the same diagonal entries 1 and 0.

Answer: $P^2 = P$.

(b) Describe the image of the entire plane and give two different inputs with the same non-zero image. [3 marks]

The coordinate rule is (x, y) to $(x, 0)$.

Answer: Image: x-axis. For example $(2, 1)$ and $(2, -3)$ both map to $(2, 0)$.

5. Two linear maps have $\det A = -2$ and $\det B = 3$. [6 marks]

(a) A region has area 5. Find its area after A then B, and state the orientation change. [3 marks]

The combined determinant is $3(-2) = -6$.

The absolute area multiplier is 6.

Answer: Area 30; orientation reverses.

(b) Find the area multiplier of the inverse combined map. [3 marks]

The combined map is invertible.

The inverse determinant is the reciprocal.

Answer: $\frac{1}{6}$.

6. Two matrices A and B are claimed to have AB of order 2×4 and BA of order 3×3. [6 marks]

(a) Show that no pair of matrices can meet both statements. [4 marks]

AB being 2×4 means A has 2 rows and B has 4 columns.

BA would therefore require 4 inner entries to match 2, which is impossible.

Answer: The dimensions are inconsistent.

(b) Give compatible orders for A and B for which AB is 2×2 and BA is 3×3. [2 marks]

The inner dimensions match in both products.

Answer: A is 2×3; B is 3×2.

7. Let $S=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. [6 marks]

(a) Find every point fixed by S and identify the transformation. [3 marks]

Solve $(y,x)=(x,y)$.

Answer: Fixed points have $y=x$; S is reflection in that line.

(b) Show that S is its own inverse and interpret the determinant. [3 marks]

Swapping coordinates twice restores them.

The determinant shows unchanged area and reversed orientation.

Answer: $S^2=I$, determinant -1.

8. Let $A=\begin{pmatrix}2 & -1\\ -4 & 2\end{pmatrix}$. [6 marks]

(a) Describe the output line and all inputs mapped to zero. [3 marks]

The second output is -2 times the first.

For zero output require $2x-y=0$.

Answer: Output line $Y=-2X$; zero inputs $(t,2t)$.

(b) Find all inputs mapped to $(3,\,-6)$. [3 marks]

The first output equation is $2x-y=3$.

The second is dependent.

Answer: $(t,2t-3)$ for real t.

9. Let R be a 90° counterclockwise rotation and $t=(a,b)$ a translation vector. [7 marks]

(a) Express the two rules: translate by t then rotate; rotate then translate. [3 marks]

Distribute R over the vector sum.

Answer: $R(p+t)=Rp+Rt$; $Rp+t$.

(b) For which t are the two rules equal for every input p? [4 marks]

Equality requires $Rt=t$.

Thus $(-b,a)=(a,b)$, giving $a=b$ and $a=-b$, hence both zero.

Answer: Only $t=(0,0)$.

10. Let $M=\begin{pmatrix}1 & k\\ 0 & -1\end{pmatrix}$. [7 marks]

(a) Show that M^2 is I for every real k. [3 marks]

The upper-right entry is $k-k=0$.

The diagonal entries are both 1.

Answer: $M^2=I$.

(b) For which k is M an ordinary orthogonal reflection in the x-axis? Justify. [4 marks]

The map fixes the x-axis.

The image of the second unit vector is $(k,-1)$, whose length is 1 only when $k=0$.

Answer: Only $k=0$.