Matrices and transformations - Practice test 3

10 questions | 61 marks | 70 minutes

Inverses and design: Choose correct inverse operations and interpret constraints, areas and whole-item resource solutions.

Equivalent correct methods earn the corresponding marks.

1. Let $A=\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}$. [6 marks]

(a) Find its determinant and inverse. [3 marks]

Use $ad-bc$ and the 2×2 inverse formula.

Answer: $\det A=3;\ A^{-1}=\frac{1}{3}\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}$.

(b) Solve $A(x,y)^T=(1,5)^T$. [3 marks]

The inverse gives $\frac{1}{3}$ of $(6,9)$.

Check $4-3=1$ and $2+3=5$.

Answer: $(x,y)=(2,3)$.

2. Let $A=\begin{pmatrix}2 & 4\\ -1 & 3\end{pmatrix}$ and $B=\begin{pmatrix}0 & 2\\ 1 & -1\end{pmatrix}$. [6 marks]

(a) Solve $2X+A=3B$. [3 marks]

Calculate $3B-A$, then halve each entry.

Answer: $X=\begin{pmatrix}-1 & 1\\ 2 & -3\end{pmatrix}$.

(b) Show that your X has an inverse and calculate it. [3 marks]

The determinant is $3-2=1$.

Apply the inverse formula.

Answer: $\det X=1;\ X^{-1}=\begin{pmatrix}-3 & -1\\ -2 & -1\end{pmatrix}$.

3. A linear map sends $(1, 0)$ to $(0, -2)$ and $(0, 1)$ to $(3, 0)$. [6 marks]

(a) Find its matrix. Is it a pure rotation? [3 marks]

Use the images as columns.

The two unit vectors have image lengths 2 and 3 rather than 1.

Answer: $A = \begin{pmatrix} 0 & 3 \\ -2 & 0 \end{pmatrix}$. It is not a pure rotation.

(b) Find the input whose image is $(9, 4)$. [3 marks]

The image rule is $(3y, -2x)$.

Thus $y = 3$ and $x = -2$.

Answer: $(-2, 3)$.

4. First reflect in $y = x$, then rotate 90° counterclockwise, then dilate by 2 about the origin. [6 marks]

(a) Find the combined matrix and give one equivalent geometric description. [4 marks]

The product is $2RF$.

Multiplication gives the stated diagonal matrix.

Answer: $T = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$; reflect in the y-axis and dilate by 2.

(b) Find the inverse matrix. [2 marks]

Undo each non-zero diagonal scale.

Answer: $T^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

5. P is projection onto the x-axis. [6 marks]

(a) Describe all inputs with image $(3, 0)$. [3 marks]

Projection keeps $x = 3$ and forgets y.

Answer: $(3, t)$ for real t.

(b) Suppose the original point is known to lie on $y = 2x + 1$. Recover it. Does this make P globally invertible? [3 marks]

Use the additional line restriction to find $y = 7$.

Without this extra information, infinitely many original points remain possible.

Answer: The point is $(3, 7)$. P remains non-invertible on the whole plane.

6. Let $M = \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}$. [7 marks]

(a) Find the values of k for which M is singular and describe the output line for each. [4 marks]

The determinant is $1 - k^2$.

At $k = 1$ the rows agree; at $k = -1$ they are opposites.

Answer: $k = 1$: line $Y = X$; $k = -1$: line $Y = -X$.

(b) Find k if the image of a unit square has area 3. [3 marks]

$|1 - k^2| = 3$ gives $k^2 = 4$ or -2.

Only k^2 equal to 4 is possible.

Answer: $k = 2$ or $k = -2$.

7. Consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. [6 marks]

(a) Explain why A preserves area but not every length. [3 marks]

Compute $2 - 1 = 1$.

A length-preserving map would send each unit vector to length 1.

Answer: Its determinant is 1, but the image of $(1, 0)$ has length $\sqrt{5}$.

(b) State the area factors of A^2 and A^{-1}. [3 marks]

Area factors multiply under composition.

The inverse has reciprocal area factor.

Answer: Both are 1.

8. A coordinate rule is $q = Rp + (4, 2)$, where R rotates 90° counterclockwise. [6 marks]

(a) Find p when $q = (5, -1)$. [3 marks]

Subtract $(4, 2)$ to get $(1, -3)$.

Rotate clockwise to recover $(-3, -1)$.

Answer: $p = (-3, -1)$.

(b) Find the image of the origin and explain why the inverse must include a translation. [3 marks]

The non-zero translation moves the origin.

An inverse rotation alone would not subtract that displacement.

Answer: Image $(4, 2)$.

9. Let $A=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$, $B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. [6 marks]

(a) Calculate AB and its inverse. [4 marks]

Multiply A and B.

Use the determinant -6 or reverse the inverse factors.

Answer: $AB=\begin{pmatrix}0 & 2\\ 3 & 0\end{pmatrix}$, $(AB)^{-1}=\begin{pmatrix}0 & \frac{1}{3}\\ \frac{1}{2} & 0\end{pmatrix}$.

(b) Is $(AB)^{-1}=A^{-1}B^{-1}$ here? Explain. [2 marks]

The correct order is $B^{-1}A^{-1}$.

$A^{-1}B^{-1}=\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{3} & 0\end{pmatrix}$, which differs.

Answer: No.

10. A workshop uses 2 units of timber and 1 of metal per small item, and 1 unit of timber and 3 of metal per large item. [6 marks]

(a) There are exactly 11 timber units and 13 metal units to use. Find the number of each item. [3 marks]

Solve $2s+l=11$ and $s+3l=13$.

Elimination gives $s=4$, then $l=3$.

Answer: 4 small and 3 large.

(b) If timber changes to 12 units while metal stays 13, can both stocks be used exactly with whole items? [3 marks]

The exact solution is $s=\frac{23}{5}$ and $l=\frac{14}{5}$.

Neither is a whole number; rounding would change resource use.

Answer: No.