Matrices and transformations - Practice test 2

10 questions | 62 marks | 70 minutes

Geometry and matrix models: Use matrices for cost changes, geometric mappings, projection, rotations and output constraints.

Equivalent correct methods earn the corresponding marks.

1. A store records notebook, folder and pen quantities in $N=\begin{pmatrix}6 & 4 & 3\\ 2 & 5 & 7\end{pmatrix}$, one row per day. Prices are $p=(4,3,2)^T$ dollars. [6 marks]

(a) Calculate Np and interpret it. [3 marks]

Day 1 costs $24+12+6=42$.

Day 2 costs $8+15+14=37$.

Answer: $(42,37)^T$ dollars.

(b) Prices change by $(1,0,-0.5)^T$ dollars. Find the change in each daily total. [3 marks]

Multiply N by the price-change vector.

The daily changes are $6-1.5$ and $2-3.5$.

Answer: $(4.5,-1.5)^T$ dollars.

2. A linear map sends $(1,1)$ to $(3,1)$ and $(1,-1)$ to $(1,3)$. [7 marks]

(a) Find the images of the two coordinate unit vectors and hence the matrix. [4 marks]

Add the supplied vector equations and divide by 2.

Subtract them and divide by 2.

Answer: $Ae_1=(2,2)^T$, $Ae_2=(1,-1)^T$; $A=\begin{pmatrix}2 & 1\\ 2 & -1\end{pmatrix}$.

(b) Find the image of $(2,3)$ and the area multiplier. [3 marks]

Multiply by A.

The determinant is -4; take its absolute value.

Answer: $(7,1)$; multiplier 4.

3. Let $F=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $D=\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$. [6 marks]

(a) Calculate FD and DF. [4 marks]

Keep factor order in the row-column products.

Answer: $FD=\begin{pmatrix}0 & 1\\ 3 & 0\end{pmatrix}$, $DF=\begin{pmatrix}0 & 3\\ 1 & 0\end{pmatrix}$.

(b) Apply both products to $(2,1)$ and explain the different geometric orders. [2 marks]

FD stretches horizontally then swaps coordinates.

DF swaps coordinates then stretches horizontally.

Answer: FD gives $(1,6)$; DF gives $(3,2)$.

4. A point is rotated 60° counterclockwise about the origin. [6 marks]

(a) Write the exact rotation matrix. [3 marks]

Substitute $\cos 60^\circ=\frac{1}{2}$ and $\sin 60^\circ=\frac{\sqrt{3}}{2}$.

Answer: $R=\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$.

(b) Find the image of $(2,0)$ and give the inverse transformation. [3 marks]

Twice the first column gives the image.

Use the opposite angle to undo the rotation.

Answer: $(1,\sqrt{3})$; rotate 60° clockwise.

5. Let P_x be projection onto the x-axis and R a 90° counterclockwise rotation. [6 marks]

(a) Find P_xR and apply it to $(3, 4)$. [3 marks]

Rotate to $(-4, 3)$, then remove the second component.

Answer: $P_xR = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$; image $(-4, 0)$.

(b) Find RP_x and apply it to the same point. Is either product invertible? [3 marks]

Project to $(3, 0)$, then rotate.

Both determinants are zero.

Answer: $RP_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; image $(0, 3)$. Neither is invertible.

6. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and solve $XA = \begin{pmatrix} 2 & 5 \\ -1 & 1 \end{pmatrix}$. [6 marks]

(a) Find X. [3 marks]

Multiply on the right by $A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

Answer: $X = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$.

(b) Calculate AX and explain why it need not equal XA. [3 marks]

Calculate the product in the new order.

Matrix multiplication is not commutative.

Answer: $AX = \begin{pmatrix} 0 & 7 \\ -1 & 3 \end{pmatrix}$.

7. A map has matrix $M = \begin{pmatrix} k & 1 \\ 2 & k \end{pmatrix}$. A region of area 3 becomes a region of area 21. [6 marks]

(a) Find every possible real k. [4 marks]

$|\det M| = 7$, so $|k^2 - 2| = 7$.

The equations give k^2 equal to 9 or -5; only 9 is possible over the reals.

Answer: $k = 3$ or $k = -3$.

(b) For each permitted k, state whether orientation is preserved. [2 marks]

Each permitted value gives determinant 7.

Answer: It is preserved for both.

8. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. [6 marks]

(a) Find every input mapped to $(5, 10)$. [3 marks]

The first image component gives $x + 2y = 5$.

The second gives the same equation multiplied by 2.

Answer: $(5 - 2t, t)$ for real t.

(b) Explain why no input maps to $(5, 11)$. [3 marks]

If $X = 5$, the second component must be 10.

Answer: All images satisfy $Y = 2X$.

9. The unit square is transformed by $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. [7 marks]

(a) Give its four image vertices in cyclic order and find its area. [4 marks]

Map the vertices $(0, 0), (1, 0), (1, 1), (0, 1)$.

The determinant is 6.

Answer: $(0, 0), (2, 1), (2, 4), (0, 3)$; area 6.

(b) Find the acute angle between the two edges meeting at the origin. [3 marks]

Use edge vectors $(2, 1)$ and $(0, 3)$, with dot product 3 and lengths $\sqrt{5}, 3$.

Answer: $\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.4^\circ$.

10. A rotation through 90° counterclockwise is about the centre $c = (1, -2)$. [6 marks]

(a) Find the image of $p = (4, -1)$. [3 marks]

Relative to c the vector is $(3, 1)$.

Rotate it to $(-1, 3)$, then add c.

Answer: $(0, 1)$.

(b) Find the input whose image is $(2, 4)$. [3 marks]

The image relative to c is $(1, 6)$.

Rotate clockwise to $(6, -1)$, then add c.

Answer: $(7, -3)$.