Matrices and transformations - Practice test 1

10 questions | 61 marks | 70 minutes

Foundations and fluency: Build from notation and arithmetic to transformations, singularity and an affine model.

Equivalent correct methods earn the corresponding marks.

1. Let $A=\begin{pmatrix}1 & 2 & 0\\ -1 & 3 & 2\end{pmatrix}$ and $B=\begin{pmatrix}2 & 1\\ 0 & -1\\ 1 & 4\end{pmatrix}$. [6 marks]

(a) State the order of A and the entry a_{23}. [2 marks]

There are two rows and three columns.

Read row 2 column 3.

Answer: 2×3; $a_{23}=2$.

(b) Calculate AB. [4 marks]

The product has order 2×2.

Dot each row of A with each column of B: 2, -1, 0 and 4.

Answer: $\begin{pmatrix}2 & -1\\ 0 & 4\end{pmatrix}$.

2. Let $C=\begin{pmatrix}2 & -1\\ 1 & 3\end{pmatrix}$, $D=\begin{pmatrix}0 & 4\\ -2 & 1\end{pmatrix}$. [6 marks]

(a) Calculate $2C-D$. [3 marks]

Double every entry of C.

Subtract matching entries of D.

Answer: $\begin{pmatrix}4 & -6\\ 4 & 5\end{pmatrix}$.

(b) Solve $3X+C=D$. [3 marks]

Rearrange to $3X=D-C$.

Subtract C and divide all entries by 3.

Answer: $X=\begin{pmatrix}-\frac{2}{3} & \frac{5}{3}\\ -1 & -\frac{2}{3}\end{pmatrix}$.

3. Let $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$. [6 marks]

(a) Calculate AB and BA. [4 marks]

Use row-column multiplication in each order.

Answer: $AB=\begin{pmatrix}2 & 3\\ 0 & 3\end{pmatrix}$, $BA=\begin{pmatrix}2 & 2\\ 0 & 3\end{pmatrix}$.

(b) Give one input vector for which the two combined transformations have different images. [2 marks]

Its images are the second columns, $(3,3)$ and $(2,3)$.

Answer: For example $(0,1)^T$.

4. Let $M=\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}$. [6 marks]

(a) Find the determinant and inverse of M. [3 marks]

Compute $3-2=1$.

Swap diagonal entries and negate off-diagonal entries.

Answer: $\det M=1$, $M^{-1}=\begin{pmatrix}1 & -1\\ -2 & 3\end{pmatrix}$.

(b) Use the inverse to solve $M(x,y)^T=(8,7)^T$. [3 marks]

Multiply the inverse by the right-hand column: $(8-7,\ -16+21)$.

Check $3+5=8$ and $2+5=7$.

Answer: $x=1$, $y=5$.

5. Consider $A = \begin{pmatrix} k & 2 \\ 3 & 6 \end{pmatrix}$. [7 marks]

(a) Find k for which A is singular. [3 marks]

The determinant is $6k - 6$.

Set it equal to zero.

Answer: $k = 1$.

(b) For $k = 1$, compare the solutions of $x + 2y = 4,\ 3x + 6y = 12$ with those of $x + 2y = 4,\ 3x + 6y = 13$. [4 marks]

The first pair has dependent consistent rows.

The second pair gives $12 = 13$ after tripling the first equation.

Answer: First: $(4 - 2t, t)$ for real t. Second: no solution.

6. A linear map sends e_1 to $(2, 1)$ and e_2 to $(-1, 2)$. [6 marks]

(a) Write its matrix. [2 marks]

Put the image vectors into columns.

Answer: $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$.

(b) Find the image of $(3, -2)$ and its area scale factor. [4 marks]

Compute $3(2, 1) - 2(-1, 2) = (8, -1)$.

The determinant is $4 + 1 = 5$.

Answer: Image $(8, -1)$; area factor 5.

7. A point is reflected in the x-axis, then rotated 90° counterclockwise about the origin. [6 marks]

(a) Find the combined matrix. [3 marks]

Call the reflection F and rotation R.

The required product is RF.

Answer: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) Find the image of $(2, -3)$ and identify the single equivalent transformation. [3 marks]

The product swaps the two components.

Answer: $(-3, 2)$; reflection in $y = x$.

8. Let P be orthogonal projection onto the y-axis. [6 marks]

(a) Write P and calculate its determinant. [3 marks]

The x-component is discarded and the y-component is retained.

Answer: $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, determinant 0.

(b) Describe every input mapped to the origin, and explain why P has no inverse. [3 marks]

All x-axis points share the zero image.

Their original x-coordinate cannot be recovered.

Answer: Every $(t, 0)$; P is not one-to-one.

9. Triangle OAB has $O=(0,0)$, $A=(4,0)$ and $B=(0,3)$. It is transformed by $T=\begin{pmatrix}2 & 1\\ 0 & -1\end{pmatrix}$. [6 marks]

(a) Find its original area and the images of A and B. [3 marks]

Original area is $\frac{1}{2}(4)(3)=6$.

Multiply the two position vectors by T.

Answer: Area 6; images $(8,0)$ and $(3,-3)$.

(b) Find the image area and say whether orientation is preserved. [3 marks]

The determinant is -2.

Multiply 6 by its absolute value 2.

Answer: Area 12; orientation reverses.

10. A camera coordinate rule is $q=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}p+(3,-1)$. [6 marks]

(a) Find p when $q=(13,3)$. [3 marks]

Subtract the translation to obtain $(10,4)$.

Apply $\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}$ to obtain $(2,4)$.

Answer: $p=(2,4)$.

(b) Can one 2×2 matrix acting alone represent this entire rule? Explain. [3 marks]

The complete rule sends the origin to $(3,-1)$.

Any matrix acting alone fixes the origin.

Answer: No.