Vectors in two dimensions - Practice test 5

10 questions | 61 marks | 75 minutes

A more demanding synthesis paper combining unknowns, modelling, projections and general proofs.

Equivalent correct methods earn the corresponding marks.

1. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$. [4 marks]

(a) Find $3\mathbf{a} + 2\mathbf{b}$ in component form. [2 marks]

$3a = (6, -3)$ and $2b = (-2, 6)$; add.

Answer: $(4, 3)$.

(b) Find a unit vector in this resultant's direction. [2 marks]

The resultant has length five, so divide by five.

Answer: $(\frac{4}{5}, \frac{3}{5})$.

2. $A = (k, 2)$, $B = (5, m)$. The displacement BA is $(-3, 4)$. [4 marks]

(a) Find k and m. [2 marks]

$BA = (k - 5, 2 - m)$; compare the two components.

Answer: $k = 2$, $m = -2$.

(b) Find the exact length of AB and a unit vector from A to B. [2 marks]

AB is the opposite displacement $(3, -4)$.

Normalise by its length five.

Answer: Length 5; unit vector $(\frac{3}{5}, -\frac{4}{5})$.

3. A non-zero vector v is parallel to $(3, 4)$, has magnitude 20 and has a negative x-component. [6 marks]

(a) Find v. [3 marks]

Write $v = t(3, 4)$; $5|t| = 20$, so $t = \pm 4$.

A negative x-component requires $t = -4$.

Answer: $(-12, -16)$.

(b) Find its direction from the positive x-axis to 0.1°. [3 marks]

It lies in quadrant III.

Add 180° to $\tan^{-1}\left(\frac{4}{3}\right) = 53.130\ldots^\circ$.

Answer: 233.1°.

4. Non-zero vectors a and b satisfy $|a| = 2$, $|b| = 3$ and $a \cdot b = 3$. [6 marks]

(a) Find their included angle. [2 marks]

The cosine is $\frac{3}{2 \times 3} = \frac{1}{2}$.

Answer: 60°.

(b) Find $|2\mathbf{a} - \mathbf{b}|$ and the scalar projection of a onto b. [4 marks]

The squared length is $4(4) - 4(3) + 9 = 13$.

The scalar projection is $\frac{a \cdot b}{|b|} = \frac{3}{3}$.

Answer: $\sqrt{13}$; 1.

5. $A=(-1,2)$, $B=(3,4)$, $C=(6,-2)$. D completes the parallelogram $ABCD$ in that order. [7 marks]

(a) Find D. [3 marks]

Use $d=a+c-b=(-1,2)+(6,-2)-(3,4)$.

Answer: $(2,-4)$.

(b) Prove that the parallelogram is a rectangle and find its area. [4 marks]

$AB=(4,2)$ and $BC=(3,-6)$ have dot product $12-12=0$.

Their lengths are $\sqrt{20}$ and $\sqrt{45}$; their product is thirty.

Answer: It is a rectangle; area 30 square units.

6. A vector u is perpendicular to $(1,2)$, and its vector projection onto the positive x-axis is $(-6,0)$. [5 marks]

(a) Find u. [3 marks]

The projection gives $x=-6$.

Perpendicularity gives $-6+2y=0$, so $y=3$.

Answer: $(-6,3)$.

(b) Find its scalar projection onto $(0,-1)$. [2 marks]

Dot $(-6,3)$ with the supplied unit direction $(0,-1)$.

Answer: -3.

7. Two adjustable forces are $m(3,4)$ N and $n(2,-2)$ N, where m,n are positive. Their resultant must be $(14,0)$ N. [7 marks]

(a) Find m and n. [4 marks]

Vertical equation: $4m-2n=0$, so $n=2m$.

Horizontal equation: $3m+2n=14$, so $7m=14$.

Answer: $m=2,\ n=4$.

(b) Compare the sum of the force magnitudes with the resultant magnitude. [3 marks]

The first force has magnitude $2(5)=10$.

The second has magnitude $4\sqrt{8}=8\sqrt{2}$.

Directions prevent magnitudes adding directly.

Answer: Sum $10+8\sqrt{2}$ N; resultant 14 N, so they are not equal.

8. Let $u=(1,t)$ and $v=(t,1)$, where t is real. [7 marks]

(a) Find all t making u and v parallel, and give the corresponding angles. [4 marks]

A scalar multiple exists when $1-t^2=0$.

At $t=1$ the vectors are equal; at $t=-1$ they are opposites.

Answer: $t=1\colon\ 0^\circ$; $t=-1\colon\ 180^\circ$.

(b) Find all t making u and v perpendicular. [3 marks]

Their dot product is $t+t=2t$.

At zero the vectors $(1,0)$ and $(0,1)$ are non-zero.

Answer: $t=0$.

9. A line through O has direction $v=(3,4)$. P has position $p=(-4,3)$. A point Q on the line has position $t\mathbf{v}$. [7 marks]

(a) Show that $|\overrightarrow{PQ}|^2=25+25t^2$. [4 marks]

Calculate $v\cdot p=-12+12=0$; both squared magnitudes are 25.

Expand the squared magnitude.

Answer: $|t\mathbf{v}-\mathbf{p}|^2=25t^2-2t(0)+25$.

(b) Hence find the closest point on the line to P and the shortest distance, without calculus. [3 marks]

Since $25t^2\geq 0$, the minimum squared distance is 25 at $t=0$.

Take its square root.

Answer: The point is O; shortest distance 5.

10. In a quadrilateral $ABCD$, M,N,P,Q are the midpoints of AB,BC,CD,DA respectively. Assume the midpoint quadrilateral is non-degenerate. [8 marks]

(a) Use position vectors to prove that $MNPQ$ is a parallelogram. [4 marks]

Write each midpoint as the average of its two endpoint position vectors.

Subtract to obtain both pairs of equal directed opposite sides.

Answer: $MN=QP=\frac{1}{2}(c-a)$ and $NP=MQ=\frac{1}{2}(d-b)$.

(b) Prove that perpendicular diagonals AC and BD make $MNPQ$ a rectangle. [4 marks]

Their dot product is one quarter of $(c-a)\cdot(d-b)=0$.

The non-degenerate parallelogram therefore has a right angle.

Answer: Adjacent side directions MN and NP are half AC and half BD.