Vectors in two dimensions - Practice test 3

10 questions | 52 marks | 70 minutes

Connect exact values, projections and geometric structure, with a short proof to finish.

Equivalent correct methods earn the corresponding marks.

1. Let $\mathbf{v} = (4\sqrt{3}, -4)$. [4 marks]

(a) Find its magnitude and direction from the positive x-axis. [2 marks]

Magnitude $\sqrt{48 + 16} = 8$.

The reference angle is 30° in quadrant IV.

Answer: 8 and 330°.

(b) Find the unit vector in the same direction. [2 marks]

Divide each component by eight.

Answer: $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$.

2. In an original coordinate system $A = (-3, 2)$ and $B = (1, 5)$. A new origin is placed at old coordinates $(2, -1)$, with axes pointing in the same directions. [5 marks]

(a) Give the new coordinates of A and B. [2 marks]

Subtract the old coordinates of the new origin from each point.

Answer: $A' = (-5, 3)$, $B' = (-1, 6)$.

(b) Calculate AB in both systems and explain the agreement. [3 marks]

Old: $(1 + 3, 5 - 2)$; new: $(-1 + 5, 6 - 3)$.

A common origin shift cancels from a difference.

Answer: $(4, 3)$ in both systems.

3. A triangular route starts with displacement $a = (2, 1)$ km, then $b = (-1, 4)$ km, then c back to the starting point. [5 marks]

(a) Find c. [2 marks]

A closed route has $a + b + c = 0$; negate $(1, 5)$.

Answer: $(-1,\ -5)$ km.

(b) Find the exact total distance travelled. [3 marks]

Add the three segment magnitudes, not the vectors.

Answer: $\sqrt{5} + \sqrt{17} + \sqrt{26}$ km.

4. Let $u = (2, 1)$, $v = (t, 3)$ and $w = (4, s)$. [4 marks]

(a) Find t if v is parallel to u. [2 marks]

The y-component multiplier is three; apply it to x.

Answer: $t = 6$.

(b) Find s if w is perpendicular to u. [2 marks]

$u \cdot w = 8 + s = 0$.

Answer: $s = -8$.

5. $A = (1, 0)$, $B = (5, 2)$, $C = (3, 6)$. [6 marks]

(a) Find angle BAC exactly. [3 marks]

$AB = (4, 2)$, $AC = (2, 6)$; dot product 20.

The cosine is $\frac{20}{\sqrt{20}\sqrt{40}} = \frac{1}{\sqrt{2}}$.

Answer: $45°$.

(b) Use a second vector calculation to classify the triangle. [3 marks]

$BA \cdot BC = (-4,\ -2) \cdot (-2, 4) = 0$.

AB and BC both have length $\sqrt{20}$.

Answer: An isosceles right-angled triangle, right angle at B.

6. A force $u = (-4, 1)$ N is resolved along the direction $v = (3, 0)$. [4 marks]

(a) Find its signed scalar projection and vector projection along v. [2 marks]

Dot product -12, direction length 3; divide for the scalar.

Multiply the unit direction $(1, 0)$ by -4.

Answer: Scalar -4 N; vector $(-4, 0)$ N.

(b) Find the perpendicular component. [2 marks]

Subtract $(-4, 0)$ from the force.

Answer: $(0, 1)$ N.

7. A line passes through O and has direction $v = (2, 1)$. P has position $u = (5, 3)$. [6 marks]

(a) Find the position of the perpendicular foot H on the line. [3 marks]

Projection coefficient is $\frac{10+3}{4+1} = \frac{13}{5}$.

Answer: $H = (\frac{26}{5}, \frac{13}{5})$.

(b) Find the exact shortest distance from P to the line. [3 marks]

PH as a difference is $(-\frac{1}{5}, \frac{2}{5})$ up to direction.

Its magnitude is $\sqrt{\frac{1}{25} + \frac{4}{25}} = \frac{1}{\sqrt{5}}$.

Answer: $\frac{1}{\sqrt{5}}$.

8. The angle between $(a, 1)$ and $(1, 1)$ is 45°. [6 marks]

(a) Form an equation involving a using the dot product. [3 marks]

Set $\frac{a+1}{\sqrt{a^2+1}\sqrt{2}} = \frac{1}{\sqrt{2}}$ and simplify.

Answer: $a + 1 = \sqrt{a^2 + 1}$.

(b) Find a and check the original angle condition. [3 marks]

Squaring gives $a^2 + 2a + 1 = a^2 + 1$, hence $a = 0$.

The original left and right sides are both one; $(0, 1)$ makes 45° with $(1, 1)$.

Answer: $a = 0$.

9. A parallelogram has adjacent sides $a = (5, 0)$ and $b = (3, 4)$. [6 marks]

(a) Show that it is a rhombus and that its diagonals are perpendicular. [3 marks]

Calculate side lengths.

Calculate $(8, 4) \cdot (2, -4) = 16 - 16$.

Answer: Both sides have length 5; diagonals $(8, 4)$ and $(2, -4)$ have dot product zero.

(b) Find its area using half the product of the diagonal lengths. [3 marks]

Diagonal lengths are $4\sqrt{5}$ and $2\sqrt{5}$.

$\text{Area} = \frac{1}{2}(4\sqrt{5})(2\sqrt{5}) = 20$.

Answer: 20 square units.

10. Non-zero vectors a and b satisfy $|a| = |b| = L$ and $a \cdot b = \frac{L^2}{2}$. O, A and B have position vectors 0, a and b. [6 marks]

(a) Prove that OAB is equilateral. [3 marks]

Every side has length L, and the included angle is 60°, so the triangle is non-degenerate.

Answer: $|b - a|^2 = L^2 + L^2 - 2(\frac{L^2}{2}) = L^2$.

(b) If M is the midpoint of AB, find the length OM. [3 marks]

$OM = \frac{a + b}{2}$.

Its squared magnitude is $\frac{L^2 + L^2 + \frac{2L^2}{2}}{4} = \frac{3L^2}{4}$.

Answer: $\frac{\sqrt{3}L}{2}$.