Vectors in two dimensions - Practice test 2

10 questions | 48 marks | 65 minutes

Use unknown components, coordinate data and direction tests to justify decisions.

Equivalent correct methods earn the corresponding marks.

1. A delivery robot visits P, Q and R in that order. Coordinates are in metres. [4 marks]

Recorded positions

Point	x (m)	y (m)
P	-2	1
Q	4	9
R	7	5

(a) Find PQ and QR. [2 marks]

Subtract each starting point from its endpoint.

Answer: $PQ = (6, 8)$; $QR = (3, -4)$.

(b) Find the total displacement PR and its exact magnitude. [2 marks]

Add the legs; magnitude is $\sqrt{9^2 + 4^2}$.

Answer: $(9, 4)$; $\sqrt{97}$ m.

2. The vectors $(2a + 1, b - 3)$ and $(7, -5)$ are equal. [5 marks]

(a) Find a and b. [2 marks]

Solve $2a + 1 = 7$ and $b - 3 = -5$.

Answer: $a = 3$, $b = -2$.

(b) Find the unit vector opposite to $(7, -5)$. [3 marks]

The length is $\sqrt{49 + 25} = \sqrt{74}$; negate and normalise.

Answer: $\left(-\frac{7}{\sqrt{74}}, \frac{5}{\sqrt{74}}\right)$.

3. Two vectors satisfy $2\mathbf{u}+\mathbf{v}=(8,1)$ and $\mathbf{u}-\mathbf{v}=(1,5)$. [5 marks]

(a) Find u and v. [3 marks]

Adding equations gives $3u=(9,6)$.

Then $v=u-(1,5)$.

Answer: $\mathbf{u}=(3,2)$, $\mathbf{v}=(2,-3)$.

(b) Find the magnitude of $3\mathbf{v}$. [2 marks]

The length of v is $\sqrt{4+9}=\sqrt{13}$; scaling by three triples it.

Answer: $3\sqrt{13}$.

4. A vector has components $(k,3)$ and magnitude 5. [4 marks]

(a) Find every possible value of k. [2 marks]

$k^2+9=25$ gives $k^2=16$.

Answer: $k=4$ or $k=-4$.

(b) Which value gives an obtuse angle with $(1,0)$? Explain. [2 marks]

The dot product with $(1,0)$ is k; it must be negative.

Answer: $k=-4$.

5. $A=(1,2)$, $B=(5,8)$, $C=(9,t)$. [5 marks]

(a) Find t so that A, B and C are collinear. [3 marks]

$AB=(4,6)$ and $AC=(8,t-2)$; matching x-components gives multiplier two.

Thus $t-2=12$.

Answer: $t=14$.

(b) For this t, state the ratio $AB:BC$ of lengths. [2 marks]

$BC=(4,6)$ equals AB, so B is the midpoint.

Answer: $1:1$.

6. Let $\mathbf{u} = (t, 1)$ and $\mathbf{v} = (2, t-3)$. [4 marks]

(a) Find t so that u and v are perpendicular. [2 marks]

Their dot product is $2t + t - 3 = 3t - 3$.

Answer: $t = 1$.

(b) For this t, find a unit vector in the direction of v. [2 marks]

$v = (2, -2)$ has length $2\sqrt{2}$; divide by that length.

Answer: $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

7. Let $\mathbf{a} = (2, 1)$ and $\mathbf{b} = (-1, 3)$. [5 marks]

(a) Find the angle between a and b to 0.1°. [3 marks]

The dot product is one; magnitude product is $\sqrt{50}$.

$\theta = \cos^{-1}(\frac{1}{\sqrt{50}})$.

Answer: 81.9°.

(b) Find the angle between a and $-b$. [2 marks]

Reversing b replaces the angle by $180^\circ - 81.869\ldots^\circ$.

Answer: 98.1°.

8. Resolve $u = (7, 1)$ relative to the direction $v = (2, 1)$. [5 marks]

(a) Find the vector projection of u onto v. [2 marks]

The coefficient is $\frac{14+1}{4+1} = 3$.

Answer: $(6, 3)$.

(b) Find the perpendicular component and its magnitude. [3 marks]

Subtract the projection: $(7, 1) - (6, 3)$.

Check its dot product with v is $2 - 2 = 0$.

Answer: $(1, -2)$ and $\sqrt{5}$.

9. A boat's velocity relative to still water is $(3, 4)$ km/h. The water current is $(1, -1)$ km/h. [5 marks]

(a) Find the boat's ground velocity and ground speed. [2 marks]

Add velocities, then use $\sqrt{16+9}=5$.

Answer: $(4, 3)$ km/h; 5 km/h.

(b) Starting from $(2, -1)$ km, where is the boat after 2 hours if both velocities remain constant? [3 marks]

Displacement is $2(4, 3)=(8, 6)$ km.

Add it to the initial position.

Answer: $(10, 5)$ km.

10. A non-degenerate parallelogram has adjacent side vectors a and b. [6 marks]

(a) Show that the squared diagonal lengths differ by $4\mathbf{a}\cdot\mathbf{b}$. [3 marks]

Expand both squared lengths with the dot product.

The squared side lengths cancel.

Answer: $|\mathbf{a}+\mathbf{b}|^2-|\mathbf{a}-\mathbf{b}|^2=4\mathbf{a}\cdot\mathbf{b}$.

(b) Hence prove that equal diagonal lengths imply the parallelogram is a rectangle. [3 marks]

Non-zero adjacent sides are therefore perpendicular.

A parallelogram with a right angle is a rectangle.

Answer: Equal lengths make the left side zero, so $a\cdot b=0$.