Further trigonometry - Practice test 3

10 questions | 42 marks | 65 minutes

Connect exact identities, restricted domains and graph information. Justify each branch of a solution set.

Equivalent correct methods earn the corresponding marks.

1. Let θ have reference angle $\frac{\pi}{3}$ in quadrant II. [4 marks]

(a) Find $\sec\theta$ and $\cot\theta$ exactly. [2 marks]

The angle is $\frac{2\pi}{3}$: cosine $-\frac{1}{2}$ and sine $\frac{\sqrt{3}}{2}$.

Take the reciprocal and ratio.

Answer: -2 and $-\frac{1}{\sqrt{3}}$.

(b) Find $\sin(\theta+\pi)$ exactly. [2 marks]

Adding π changes the sine sign: $\sin(\theta+\pi)=-\sin\theta$.

Answer: $-\frac{\sqrt{3}}{2}$.

2. A sinusoidal signal has maximum 9 and minimum -3. It crosses its midline upwards at $x=1$, and its next upward midline crossing is at $x=5$. [4 marks]

(a) Construct a sine rule. [2 marks]

Midline 3 and amplitude 6; period 4 gives frequency $\frac{\pi}{2}$.

An upward midline crossing selects positive sine at argument zero.

Answer: $y=3+6\sin(\frac{\pi}{2}(x-1))$.

(b) Find its first maximum for $x\geq 1$. [2 marks]

A quarter period after the upward crossing is 1 unit later.

Answer: At $x=2$, with value 9.

3. Solve $2\sin^2 x + \sin x - 1 = 0$ on $[0, 2\pi)$. [5 marks]

(a) Factor the equation and find every angle. [3 marks]

$(2\sin x - 1)(\sin x + 1) = 0.$

Sine $\frac{1}{2}$ gives the first two; sine -1 gives $\frac{3\pi}{2}$.

Answer: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

(b) Explain why neither algebraic root is rejected before finding angles. [2 marks]

A substituted sine value must be in $[-1, 1]$.

Answer: Both $\frac{1}{2}$ and -1 belong to the sine range.

4. Use $105^\circ = 60^\circ + 45^\circ$. [4 marks]

(a) Find $\cos 105^\circ$ exactly. [2 marks]

Compute $(\frac{1}{2})(\frac{\sqrt{2}}{2}) - (\frac{\sqrt{3}}{2})(\frac{\sqrt{2}}{2})$.

Answer: $\frac{\sqrt{2} - \sqrt{6}}{4}$.

(b) Find $\tan 105^\circ$ exactly. [2 marks]

The angle is $180^\circ - 75^\circ$, so tangent is $-\tan 75^\circ$.

Alternatively rationalise the tangent-sum formula.

Answer: $-(2 + \sqrt{3})$.

5. An angle x is acute and $\cos 2x = \frac{7}{25}$. [4 marks]

(a) Find $\sin x$ and $\cos x$ exactly. [2 marks]

Use sine square $\frac{1 - \frac{7}{25}}{2} = \frac{9}{25}$ and cosine square $\frac{1 + \frac{7}{25}}{2} = \frac{16}{25}$.

Choose positive roots because x is acute.

Answer: $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$.

(b) Find $\sin 4x$. [2 marks]

First $\sin 2x = 2(\frac{3}{5})(\frac{4}{5}) = \frac{24}{25}$.

Then $\sin 4x = 2(\frac{24}{25})(\frac{7}{25})$.

Answer: $\frac{336}{625}$.

6. Consider $w(x) = 7\sin x - 24\cos x$. [4 marks]

(a) Find its amplitude and express it as $R\sin(x+\phi)$ with $-\frac{\pi}{2} < \phi < 0$. [2 marks]

Match $\cos\phi = \frac{7}{25}$ and $\sin\phi = -\frac{24}{25}$.

Answer: $R = 25$, $\phi = -\tan^{-1}\left(\frac{24}{7}\right)$.

(b) Find its maximum and the first nonnegative input where it occurs. [2 marks]

A maximum requires $x + \phi = \frac{\pi}{2} + 2k\pi$; choose $k = 0$.

Answer: Maximum 25 at $x = \frac{\pi}{2} + \tan^{-1}\left(\frac{24}{7}\right)$.

7. Consider $r(x) = 3\csc(2x) - 1$. [4 marks]

(a) State the period and domain. [2 marks]

Cosecant has period 2π in its argument; its denominator is zero at $2x = k\pi$.

Answer: Period π; exclude $x = \frac{k\pi}{2}$.

(b) State its range. [2 marks]

Scale $(-\infty, -1] \cup [1, \infty)$ by 3 and subtract 1.

Answer: $(-\infty, -4] \cup [2, \infty)$.

8. A student simplifies $\frac{\sin 2x}{1 - \cos 2x}$ to $\cot x$. [5 marks]

(a) Show the simplification with restrictions. [3 marks]

Use $\frac{2\sin x\cos x}{2\sin^2 x}$; require $\sin x \neq 0$ before cancelling.

Answer: $\cot x$ with $x \neq k\pi$.

(b) Does the original expression equal zero at $x = \frac{\pi}{2}$? Explain. [2 marks]

Numerator is $\sin\pi = 0$ and denominator is $1 - \cos\pi = 2$.

Answer: Yes, it is 0.

9. A rotating beacon has model brightness $B(t) = 4 + 3\sin(\frac{\pi t}{5})$, in arbitrary units, for $0 \le t \le 10$ seconds. [4 marks]

(a) Find when brightness equals $\frac{11}{2}$. [2 marks]

Sine equals $\frac{1}{2}$ at argument $\frac{\pi}{6}, \frac{5\pi}{6}$.

Multiply by $\frac{5}{\pi}$.

Answer: $t = \frac{5}{6}, \frac{25}{6}$ s.

(b) Find the duration for which brightness exceeds $\frac{11}{2}$. [2 marks]

Between the two crossing times: $\frac{25}{6} - \frac{5}{6} = \frac{10}{3}$.

Answer: $\frac{10}{3}$ s.

10. An expression $a\sin x + b\cos x$ has amplitude 5 and is equal to 3 at $x = 0$. [4 marks]

(a) Find the possible pairs (a, b). [2 marks]

At 0 the expression equals $b = 3$.

Amplitude gives $a^2 + 9 = 25$, so $a = \pm 4$.

Answer: $(4, 3)$ or $(-4, 3)$.

(b) What extra observation would distinguish the two possibilities without differentiation? [2 marks]

At $\frac{\pi}{2}$ the value equals a, so it is 4 or -4.

Answer: Its value at $x = \frac{\pi}{2}$.