Proof and circle geometry - Challenge test

Name: _____ Date: _____

10 questions | 61 marks | 75 minutes

A final synthesis paper requiring careful domains, valid converses and multi-step geometric reasoning.

No calculator needed.

1. A rectangle has positive side lengths a and b. [7 marks]

(a) Prove its two diagonals have equal lengths. [3 marks]

(b) Is the converse “equal diagonals imply a rectangle” true for all quadrilaterals? Give a checked counterexample. [4 marks]

2. Work over the integers. [6 marks]

(a) Prove an odd integer has an odd square. [2 marks]

(b) Hence prove the equivalence: n is even if and only if n^2 is even. [4 marks]

3. Investigate operations on irrational numbers. [6 marks]

(a) Give two irrational numbers whose sum and product are both rational. [2 marks]

(b) Give two irrational numbers with rational sum but irrational product, and justify both claims. [4 marks]

4. Compare two number systems. [6 marks]

(a) Prove there is no least positive rational number. [3 marks]

(b) Explain why this argument does not show there is no least positive integer. [3 marks]

5. Solve $\frac{x-1}{x+2} \geq 0$ for real x. [6 marks]

(a) Identify the values where the sign can change and the excluded input. [2 marks]

(b) Use a sign analysis to write the full interval solution. [4 marks]

6. For chord AB, a minor central angle is $3x+6$ degrees. An inscribed angle on the major arc AB is $x+13$ degrees. [5 marks]

(a) Find x and both angles. [3 marks]

(b) Find the inscribed angle from a point on the minor arc AB. [2 marks]

7. $ABCD$ is cyclic in that order. Its opposite angles A and C are $x+30$ and $2x$ degrees. Its opposite angles B and D are $2y-10$ and $y+40$ degrees. [7 marks]

(a) Find x, y and all four interior angles. [4 marks]

(b) Which diagonal is a diameter? Justify it. [3 marks]

8. A triangle ABC is inscribed in a circle. Angle $BAC = 58^\circ$ and angle $ACB = 36^\circ$. A tangent is drawn at A. [5 marks]

(a) Find angle ABC. [2 marks]

(b) Find the acute angle between the tangent and chord AC, naming the relevant theorem. [3 marks]

9. A tangent from P is 6 cm long. A secant from P has outside length x cm and inside length 9 cm. [6 marks]

(a) Find x, rejecting any invalid root. [3 marks]

(b) A second secant from P has outside length 4 cm. Find its inside length. [3 marks]

10. Distinct points A, B, C, D lie in that order on a circle. Chords AC and BD intersect at E inside the circle. [7 marks]

(a) Prove triangles AEB and DEC are similar and deduce $AE \cdot EC = BE \cdot ED$. [5 marks]

(b) If $AE = 6$ cm, $EC = 10$ cm and $BE = 4$ cm, find ED. [2 marks]