Proof and circle geometry - Challenge test

10 questions | 61 marks | 75 minutes

A final synthesis paper requiring careful domains, valid converses and multi-step geometric reasoning.

Equivalent correct methods earn the corresponding marks.

1. A rectangle has positive side lengths a and b. [7 marks]

(a) Prove its two diagonals have equal lengths. [3 marks]

Each diagonal is the hypotenuse of a right triangle with legs a and b.

Answer: Both diagonals have length $\sqrt{a^2+b^2}$.

(b) Is the converse “equal diagonals imply a rectangle” true for all quadrilaterals? Give a checked counterexample. [4 marks]

Take vertices $(0,0),(4,0),(3,2),(1,2)$. Both diagonals have length $\sqrt{13}$, but the sloping sides are not perpendicular to the horizontal bases.

Answer: No.

2. Work over the integers. [6 marks]

(a) Prove an odd integer has an odd square. [2 marks]

$(2k+1)^2 = 2(2k^2+2k)+1.$

Answer: It does.

(b) Hence prove the equivalence: n is even if and only if n^2 is even. [4 marks]

If $n=2k$ then $n^2=2(2k^2)$ is even.

If n^2 is even but n were odd, part (a) would make n^2 odd, a contradiction.

Answer: Both directions hold.

3. Investigate operations on irrational numbers. [6 marks]

(a) Give two irrational numbers whose sum and product are both rational. [2 marks]

Their sum is 0 and product -2.

Answer: $\sqrt{2}$ and $-\sqrt{2}$.

(b) Give two irrational numbers with rational sum but irrational product, and justify both claims. [4 marks]

Both are irrational; their sum is 2.

The product is $2\sqrt{2}-2$, irrational because a rational value would make $\sqrt{2}$ rational.

Answer: $\sqrt{2}$ and $2-\sqrt{2}$.

4. Compare two number systems. [6 marks]

(a) Prove there is no least positive rational number. [3 marks]

If q were least and positive, $\frac{q}{2}$ would be a smaller positive rational, a contradiction.

Answer: No such rational exists.

(b) Explain why this argument does not show there is no least positive integer. [3 marks]

In particular, $\frac{1}{2}$ is not an integer. The positive integers do have least member 1.

Answer: Dividing an integer by 2 need not leave an integer.

5. Solve $\frac{x-1}{x+2} \geq 0$ for real x. [6 marks]

(a) Identify the values where the sign can change and the excluded input. [2 marks]

The numerator is zero at 1 and denominator at -2.

Answer: Critical values -2 and 1; exclude -2.

(b) Use a sign analysis to write the full interval solution. [4 marks]

For $x<-2$ both factors are negative, giving a positive ratio.

For $-2<x<1$ the ratio is negative; for $x>1$ it is positive. Include zero at $x=1$, but never $x=-2$.

Answer: $(-\infty, -2) \cup [1, \infty)$.

6. For chord AB, a minor central angle is $3x+6$ degrees. An inscribed angle on the major arc AB is $x+13$ degrees. [5 marks]

(a) Find x and both angles. [3 marks]

$3x+6=2(x+13)$ gives $x=20$.

Answer: $x=20$; central 66°, inscribed 33°.

(b) Find the inscribed angle from a point on the minor arc AB. [2 marks]

It intercepts the major arc of $360-66=294$ degrees; halve to get 147.

Answer: 147°.

7. $ABCD$ is cyclic in that order. Its opposite angles A and C are $x+30$ and $2x$ degrees. Its opposite angles B and D are $2y-10$ and $y+40$ degrees. [7 marks]

(a) Find x, y and all four interior angles. [4 marks]

$3x+30=180$ and $3y+30=180$. Substitute to find each angle.

Answer: $x=y=50$; $A=80$, $B=90$, $C=100$, $D=90$ degrees.

(b) Which diagonal is a diameter? Justify it. [3 marks]

Angle ABC is 90°, so the arc AC not containing B is a semicircle; AC is a diameter.

Answer: AC.

8. A triangle ABC is inscribed in a circle. Angle $BAC=58^\circ$ and angle $ACB=36^\circ$. A tangent is drawn at A. [5 marks]

(a) Find angle ABC. [2 marks]

$180-58-36=86$.

Answer: 86°.

(b) Find the acute angle between the tangent and chord AC, naming the relevant theorem. [3 marks]

The alternate-segment theorem matches that tangent-chord angle with angle ABC.

Answer: 86°.

9. A tangent from P is 6 cm long. A secant from P has outside length x cm and inside length 9 cm. [6 marks]

(a) Find x, rejecting any invalid root. [3 marks]

$x(x+9)=36$, so $(x+12)(x-3)=0$. Reject -12 as a length.

Answer: $x=3$ cm.

(b) A second secant from P has outside length 4 cm. Find its inside length. [3 marks]

Its whole length L satisfies $4L=36$, so $L=9$ and inside length is $9-4=5$.

Answer: 5 cm.

10. Distinct points A,B,C,D lie in that order on a circle. Chords AC and BD intersect at E inside the circle. [7 marks]

(a) Prove triangles AEB and DEC are similar and deduce $AE\cdot EC=BE\cdot ED$. [5 marks]

Angles AEB and DEC are vertically opposite.

Angles ABE and DCE equal angles ABD and DCA; both subtend chord AD in the same segment.

Thus AEB corresponds to DEC. From $\frac{AE}{DE}=\frac{BE}{CE}$, cross-multiply.

Answer: They are similar by AA, giving the stated product.

(b) If $AE=6$ cm, $EC=10$ cm and $BE=4$ cm, find ED. [2 marks]

$6\times10=4\times ED$, so $ED=15$.

Answer: 15 cm.