# Proof and circle geometry - Applications test

10 questions | 56 marks | 75 minutes

Use logical claims, exact arithmetic and circle theorems to analyse practical designs and identify errors.

Equivalent correct methods earn the corresponding marks.

**1.** A numerical checker accepts an integer exactly when it is divisible by both 3 and 5. [5 marks]

**(a)** Express the rule as an equivalence using divisibility by one integer. [3 marks]

3 and 5 are distinct primes, so their product divides any integer divisible by both. A multiple of 15 is visibly divisible by both.

**Answer:** Accepted $\Leftrightarrow$ divisible by 15.

**(b)** Does acceptance imply divisibility by 30? Give evidence. [2 marks]

15 is accepted but is not divisible by 30.

**Answer:** No.

**2.** An algorithm assumes $a < b$ always implies $a^2 < b^2$. [5 marks]

**(a)** Give a counterexample using negative integers. [2 marks]

$-3 < -2$ but $9 > 4$.

**Answer:** $a = -3,\ b = -2$.

**(b)** Prove the implication is valid under the extra condition $0 \leq a < b$. [3 marks]

$b^2 - a^2 = (b - a)(b + a)$. Both factors are positive, so $b^2 > a^2$.

**Answer:** It is valid on this domain.

**3.** A student says: “If $x+y$ is rational, then $x$ and $y$ must each be rational.” [5 marks]

**(a)** Disprove this implication. [2 marks]

Both terms are irrational but their sum is 0.

**Answer:** $x=\sqrt{2}$, $y=-\sqrt{2}$.

**(b)** Prove that if $x$ is rational and $x+y$ is rational, then $y$ is rational. [3 marks]

$y=(x+y)-x$ is the difference of two rational numbers.

**Answer:** It follows by subtraction.

**4.** Prove that $\sqrt{3}$ is irrational. [7 marks]

**(a)** Show an integer not divisible by 3 has a square not divisible by 3. [3 marks]

The integer is $3k+1$ or $3k+2$; squaring gives remainder 1 modulo 3 in either case.

**Answer:** Its square has remainder 1.

**(b)** Use a fraction in lowest terms to complete a contradiction proof. [4 marks]

Assume $\sqrt{3}=\frac{a}{b}$ in lowest terms. Then $a^2=3b^2$, so part (a) forces $a=3k$.

Substitution gives $b^2=3k^2$, so 3 divides $b$ as well, contradicting lowest terms.

**Answer:** No rational fraction equals $\sqrt{3}$.

**5.** A formula $f(x)=\frac{1}{x-2}$ is to be used only when $-1\le x<5$. [5 marks]

**(a)** Write its permitted input set in interval notation. [3 marks]

Exclude $x=2$ because its denominator is zero; keep the original outside bounds.

**Answer:** $[-1,2)\cup(2,5)$.

**(b)** Explain why $[-1,5)$ alone is incorrect. [2 marks]

At $x=2$ the formula is undefined.

**Answer:** It includes the forbidden input 2.

**6.** Two camera positions $P$ and $Q$ lie on opposite segments of chord $AB$ of a circular arena. The minor central angle $AOB$ is $84^\circ$. $P$ lies on the major arc and $Q$ on the minor arc. [5 marks]

**(a)** Find the viewing angle $APB$. [2 marks]

$P$ intercepts the minor arc: $\frac{84}{2} = 42$.

**Answer:** $42^\circ$.

**(b)** Find $AQB$ and explain why the two angles differ. [3 marks]

$Q$ intercepts the major arc of $276^\circ$; halve it to get $138^\circ$. The vertices lie in opposite segments.

**Answer:** $138^\circ$.

**7.** A four-sided frame $ABCD$ is designed to fit a circular opening. In cyclic order its angles $A$ and $C$ are $2x + 10$ and $3x - 5$ degrees, and $B$ is $110^\circ$. [6 marks]

**(a)** Find $x$, $A$ and $C$. [3 marks]

$2x + 10 + 3x - 5 = 180$ gives $5x = 175$.

**Answer:** $x = 35$; $A = 80^\circ$, $C = 100^\circ$.

**(b)** Find the exterior angle at $D$ formed by extending one side. [3 marks]

$D = 180 - 110 = 70^\circ$; its exterior angle is $180 - 70 = 110^\circ$, equal to opposite $B$.

**Answer:** $110^\circ$.

**8.** A circular pond has radius 9 m. A post $P$ is 15 m from its centre. Two straight cables from $P$ just touch the pond at $A$ and $B$. [5 marks]

**(a)** Find the length of cable $PA$. [3 marks]

The radius to $A$ is perpendicular to $PA$. $PA = \sqrt{15^2 - 9^2} = 12$.

**Answer:** 12 m.

**(b)** Find the total length of the two cables and justify the equality you use. [2 marks]

Tangents $PA$ and $PB$ from the same external point are equal, so the total is $2 \times 12$.

**Answer:** 24 m.

**9.** A survey line from $P$ crosses a circle at $A$ and $B$, with $PA = 6$ m and $AB = 18$ m. A tangent from $P$ touches at $T$. [6 marks]

**(a)** Find $PT$. [3 marks]

$PB = 24$; $PT^2 = 6 \times 24 = 144$.

**Answer:** 12 m.

**(b)** A second survey line has outside portion 9 m. Find its portion inside the circle. [3 marks]

The full length $L$ satisfies $9L = 144$, giving $L = 16$; subtract 9.

**Answer:** 7 m.

**10.** A circular skylight has centre $O$ and a non-diameter chord $AB$. $M$ is the midpoint of $AB$. [7 marks]

**(a)** Prove $OM$ is perpendicular to $AB$. [4 marks]

$OA = OB$ are radii, $AM = BM$ from midpoint, and $OM$ is shared.

The equal angles $OMA$ and $OMB$ lie on a straight line, so both are $90^\circ$.

**Answer:** Triangles $OMA$ and $OMB$ are congruent by SSS.

**(b)** If the radius is 10 cm and $AB$ is 16 cm, find $OM$. [3 marks]

$AM = 8$ and $OM^2 = 10^2 - 8^2 = 36$.

**Answer:** 6 cm.