Proof and circle geometry - Practice test 2

10 questions | 52 marks | 75 minutes

A second balanced paper emphasising converse arguments, divisibility, constructions and similarity.

Equivalent correct methods earn the corresponding marks.

1. For a real number x, consider the statement $x = 3 \Rightarrow x^2 = 9$. [5 marks]

(a) Explain why the implication is true. [2 marks]

The conclusion follows from the stated value of x.

Answer: Substitution gives $3^2 = 9$.

(b) Write a corrected equivalence for $x^2 = 9$. [3 marks]

Factor $(x - 3)(x + 3) = 0$ and use the zero-product property.

Answer: $x^2 = 9 \Leftrightarrow x = 3$ or $x = -3$.

2. Consider two geometric claims. [5 marks]

(a) Disprove: every quadrilateral with a pair of parallel sides is a parallelogram. [3 marks]

For example, join $(0, 0), (4, 0), (3, 2), (1, 2)$ in order. The horizontal sides are parallel but the sloping sides are not.

Answer: A non-parallelogram trapezium is a counterexample.

(b) Disprove: every rhombus is a square. [2 marks]

It has four equal sides but no right angles.

Answer: A rhombus with angles 60° and 120°.

3. Let n be an integer. [5 marks]

(a) Prove $n(n+1)$ is even. [2 marks]

Consecutive integers cannot both be odd; one has a factor of 2.

Answer: It is even.

(b) Prove $n(n+1)(n+2)$ is divisible by 6. [3 marks]

Among three consecutive integers one is divisible by 3, and at least one is even.

Thus their product has factors 2 and 3, and is divisible by 6.

Answer: It is divisible by 6.

4. Let r be a non-zero rational number and x be irrational. [5 marks]

(a) Prove rx is irrational. [3 marks]

If rx were rational, dividing it by non-zero rational r would make x rational, a contradiction.

Answer: It is irrational.

(b) Explain why the condition $r \neq 0$ is essential. [2 marks]

The conclusion fails when the rational multiplier is zero.

Answer: If $r = 0$, $rx = 0$ is rational.

5. Describe solution sets over the reals. [4 marks]

(a) Solve $x^2 \leq 16$ and write an interval. [2 marks]

The distance of x from zero is at most 4; equality is allowed.

Answer: $[-4, 4]$.

(b) Find $[-4, 4] \cap (2, \infty)$. [2 marks]

Both conditions hold only for x greater than 2 and at most 4.

Answer: $(2, 4]$.

6. AB is a diameter and P is another point on its circle. AP is 7 cm and PB is 24 cm. [5 marks]

(a) Find AB. [3 marks]

Angle APB is right by the semicircle theorem. $AB = \sqrt{7^2 + 24^2} = 25$.

Answer: 25 cm.

(b) A student calls AP the hypotenuse. Explain the error using the angle location. [2 marks]

The hypotenuse joins the other two vertices of a right triangle.

Answer: AB is opposite the right angle at P, so AB is the hypotenuse.

7. $ABCD$ is an isosceles trapezium with AB parallel to CD, and base angles A and B equal. [5 marks]

(a) Explain why angle A plus angle D is 180°. [2 marks]

AB is parallel to CD.

Answer: They are co-interior angles on transversal AD.

(b) Prove the trapezium is cyclic. [3 marks]

$A = B$ by hypothesis, and $A + D = 180$, hence $B + D = 180$.

B and D are opposite; use the converse cyclic theorem.

Answer: Angles B and D are supplementary, so $ABCD$ is cyclic.

8. Tangents PA and PB touch a circle with centre O. The angle APB is 70°. [6 marks]

(a) Find angle AOB. [3 marks]

Angles OAP and OBP are right. The quadrilateral angle sum gives $AOB = 360 - 90 - 90 - 70 = 110$.

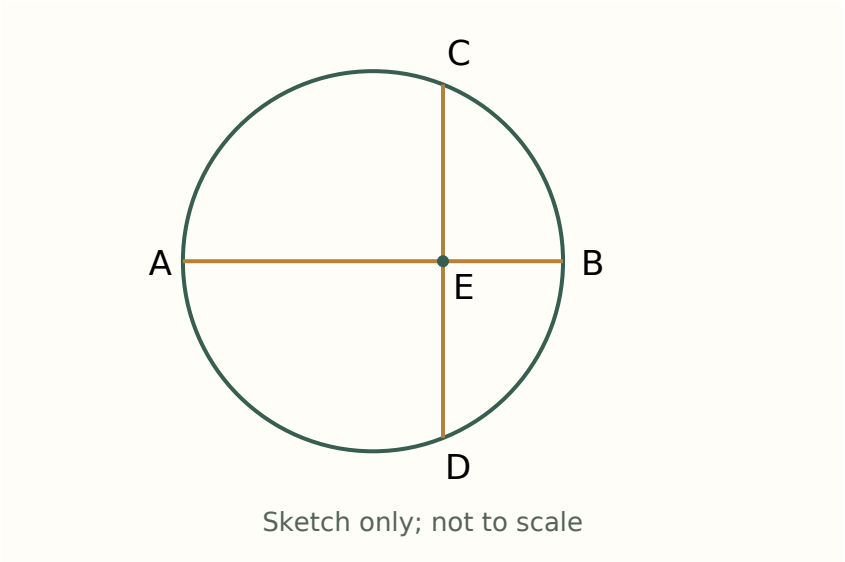
Answer: 110°.

(b) Explain why OP bisects angle APB. [3 marks]

$OA = OB$, OP is shared and the contact angles are right, so the angles at P are equal.

Answer: Right triangles OAP and OBP are congruent by RHS.

9. Chords AB and CD meet at E. $AE = x$, $EB = x + 5$, $CE = 3$ and $ED = 12$, in centimetres. [5 marks]



Sketch only; not to scale

(a) Form and solve an equation for x. [3 marks]

$x(x + 5) = 36$ gives $x^2 + 5x - 36 = (x + 9)(x - 4) = 0$.

Answer: $x = 4$ or $x = -9$ algebraically.

(b) State the permitted x and find AB. [2 marks]

A segment length is positive, so reject -9. $AB = x + (x + 5) = 13$.

Answer: $x = 4$ cm and $AB = 13$ cm.

10. In a right triangle ABC, angle A is right and altitude AH meets BC. $BH = 9$ cm and $HC = 16$ cm. [7 marks]

(a) Find AH using the geometric-mean relationship. [3 marks]

AA similarity gives $AH^2 = BH \cdot HC = 144$.

Answer: 12 cm.

(b) Find AB and AC, explaining which hypotenuse piece belongs to each. [4 marks]

$BC = 25$. Similarity gives $AB^2 = BH \cdot BC = 225$ and $AC^2 = HC \cdot BC = 400$.

Answer: $AB = 15$ cm; $AC = 20$ cm.