Proof and circle geometry - Practice test 1

10 questions | 48 marks | 75 minutes

A balanced paper moving from logical statements and exact number arguments to circle geometry and a final congruence proof.

Equivalent correct methods earn the corresponding marks.

1. Work over the integers. Consider: if n is a multiple of 8, then n is even. [4 marks]

(a) Prove the implication. [2 marks]

Write $n = 8k = 2(4k)$ with $k \in \mathbb{Z}$.

Answer: True.

(b) Write its converse and disprove that converse. [2 marks]

2 is even but not divisible by 8.

Answer: If n is even, then n is a multiple of 8; false at $n = 2$.

2. A student claims $x^2 \geq x$ for every real x. [4 marks]

(a) Find a counterexample and show the failed inequality. [2 marks]

$x^2 = \frac{1}{4} < \frac{1}{2}$.

Answer: $x = \frac{1}{2}$.

(b) Explain why testing $x = 1, 2, 3$ does not prove the original claim. [2 marks]

A universal claim needs a general argument, while one admissible counterexample disproves it.

Answer: Those tests do not cover all real numbers.

3. Let a and b be odd integers. [6 marks]

(a) Prove their sum is even. [3 marks]

Write $a = 2r + 1$, $b = 2s + 1$. Then $a + b = 2(r + s + 1)$.

Answer: It is even.

(b) Prove their product is odd. [3 marks]

$ab = 4rs + 2r + 2s + 1 = 2(2rs + r + s) + 1$.

Answer: It is odd.

4. Use exact reasoning about rational numbers. [5 marks]

(a) Show $0.\overline{36}$ is rational. [2 marks]

Set x equal to the recurring decimal. $100x - x = 36$, so $x = \frac{36}{99} = \frac{4}{11}$.

Answer: $\frac{4}{11}$.

(b) Given that $\sqrt{3}$ is irrational, prove $2 - \sqrt{3}$ is irrational. [3 marks]

If $2 - \sqrt{3} = r$ were rational, then $\sqrt{3} = 2 - r$ would be rational, a contradiction.

Answer: It is irrational.

5. Let $A = [-3, 4)$ and $B = (1, 7]$. [4 marks]

(a) Find $A \cap B$. [2 marks]

A point must exceed 1 and be less than 4. Neither endpoint lies in both sets.

Answer: $(1, 4)$.

(b) Find $A \cup B$. [2 marks]

The intervals overlap; the outside endpoints -3 and 7 are included.

Answer: $[-3, 7]$.

6. A and B are on a circle with centre O. The minor central angle AOB is 118°. P is on the major arc AB, and Q is on the minor arc AB. [4 marks]

(a) Find angle APB. [2 marks]

P intercepts the minor arc: $\frac{118}{2} = 59$.

Answer: 59°.

(b) Find angle AQB and explain the different arc. [2 marks]

Q intercepts the major arc of $360 - 118 = 242$ degrees, giving $\frac{242}{2} = 121$.

Answer: 121°.

7. $ABCD$ is a cyclic quadrilateral in that order. Angle A is $3x + 5$ degrees and angle C is $2x + 25$ degrees. Angle B is 96°. [4 marks]

(a) Find x and angle A. [2 marks]

$3x + 5 + 2x + 25 = 180$, hence $5x = 150$ and $x = 30$.

Answer: $x = 30$; $A = 95^\circ$.

(b) Find angle D and check all four angles sum to a full turn. [2 marks]

$C = 85^\circ$ and $D = 180 - 96 = 84^\circ$.

$95 + 96 + 85 + 84 = 360$.

Answer: $D = 84^\circ$; angle sum 360°.

8. A chord is 8 cm from the centre of a circle of radius 17 cm. [5 marks]

(a) Find the length of the chord. [3 marks]

The perpendicular bisects it. Half-length $a = \sqrt{17^2 - 8^2} = 15$; double to get 30.

Answer: 30 cm.

(b) A second chord is also 30 cm long. State its distance from the centre with a reason. [2 marks]

Equal chords in the same circle are equally distant from the centre.

Answer: 8 cm.

9. From an external point P, a secant meets a circle at near point A and far point B. PA is 5 cm and AB is 15 cm. PT is tangent. [6 marks]

(a) Find PT. [3 marks]

The whole secant PB is $5 + 15 = 20$ cm. $PT^2 = PA \cdot PB = 100$.

Answer: 10 cm.

(b) Another secant has outside segment 4 cm. Find its inside segment. [3 marks]

Its whole length L satisfies $4L = 100$, so $L = 25$. Subtract the outside 4 to get 21.

Answer: 21 cm.

10. Triangle ABC has $AB = AC$. D is the midpoint of BC. [6 marks]

(a) Prove triangles ABD and ACD congruent, matching the vertices. [3 marks]

$AB = AC$ is given, $BD = CD$ follows from midpoint, and AD is shared.

Answer: $\triangle ABD \cong \triangle ACD$ by SSS.

(b) Deduce that AD bisects angle BAC and is perpendicular to BC. [3 marks]

Angles BAD and DAC are equal by congruence.

Angles ADB and ADC are equal and supplementary, so each is 90°.

Answer: Both conclusions follow from corresponding angles.