Marking guide | Continuous probability and normal distributions

Practice test 6 - connected applications

10 questions | 47 marks | 75 minutes

A more demanding paper joining exact density work, normal parameter inference and independent-sample reasoning.

Equivalent correct methods earn the corresponding marks.

1. A model has density $f(x) = k(1 - x^2)$ for $-1 \le x \le 1$, zero elsewhere. [5 marks]

(a) Find k and justify nonnegativity. [2 marks]

$\int_{-1}^{1}(1 - x^2)\,dx = \frac{4}{3}$, so $k = \frac{3}{4}$. On the support, $1 - x^2 \ge 0$.

Answer: $k = \frac{3}{4}$.

(b) Find $P(|X| < 0.5)$. [3 marks]

By symmetry, integrate $2(\frac{3}{4})(1 - x^2)$ from 0 to $\frac{1}{2}$, giving $\frac{3}{4} - \frac{1}{16} = \frac{11}{16}$.

Answer: $\frac{11}{16}$.

2. For the density in Question 1, [5 marks]

(a) find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$. [3 marks]

Symmetry gives mean zero.

$\mathrm{E}(X^2) = \frac{3}{4}\int_{-1}^{1}(x^2 - x^4)\,dx = \frac{1}{5}$.

Answer: 0 and $\frac{1}{5}$.

(b) find the mean and variance of $Y = 4X + 7$. [2 marks]

Use the linear transformation rules.

Answer: 7 and $\frac{16}{5}$.

3. For the same model, [5 marks]

(a) show that its interior CDF is $\frac{1}{2}+\frac{3}{4}x-\frac{1}{4}x^3$. [3 marks]

Integrate $(\frac{3}{4})(1-t^2)$ from -1 to x and simplify.

Answer: $F(x)=\frac{1}{2}+\frac{3x}{4}-\frac{x^3}{4}$.

(b) Use symmetry to find the median, then explain why the 90th percentile must be positive. [2 marks]

$F(0)=0.5$; a 0.90 cumulative area lies to the right of the median.

Answer: Median 0; the 90th percentile is positive.

4. A normal model has its 25th percentile at 42 and its 75th percentile at 58. [5 marks]

(a) Find its mean and SD. [3 marks]

Symmetry gives mean 50.

$58=50+\Phi^{-1}(0.75)\sigma$.

Answer: Mean 50; SD 11.8608.

(b) Find its probability of lying below 30. [2 marks]

Standardise 30 with the unrounded inferred SD.

Answer: 0.0459

5. For a normal variable with mean 100, $P(X>112)=0.10$. [5 marks]

(a) Find its SD. [2 marks]

The upper-tail condition makes 112 the 90th percentile.

Answer: 9.3636.

(b) Find a symmetric interval about 100 containing 98% of values. [3 marks]

Use the 1st and 99th percentiles with the same unrounded SD.

Answer: $[78.2169, 121.7831]$.

6. Independent $X \sim N(50, 16)$ and $Y \sim N(52, 9)$ model two independent measurements. [5 marks]

(a) Find the distribution of $X - Y$. [2 marks]

Mean is -2; variance is $16 + 9 = 25$.

Answer: $N(-2, 25)$.

(b) Find the probability that the two measurements differ by less than 3. [3 marks]

The event is $-3 < X - Y < 3$. Standardise using mean -2 and SD 5, giving -0.2 and 1.

Answer: 0.4206.

7. An independent random sample is taken from a normal population with SD 12. [4 marks]

(a) Find the smallest sample size making the standard error at most 2. [2 marks]

$\frac{12}{\sqrt{n}} \leq 2$ implies $n \geq 36$.

Answer: 36.

(b) For that sample size and population mean 80, find $P(\text{sample mean} > 83)$. [2 marks]

The standard error is 2, giving $z = 1.5$.

Answer: 0.0668

8. An independent random sample of 64 is taken from a non-normal population with mean 40 and finite SD 8. [4 marks]

(a) Approximate $P(38 < \text{sample mean} < 41)$. [2 marks]

$\text{SE} = 1$. Apply a CLT approximation with z-bounds -2 and 1.

Answer: 0.8186.

(b) State two limitations of interpreting this probability. [2 marks]

The mean and SD alone do not guarantee a good finite-sample normal approximation.

Answer: The sample must be suitably random/independent, and the normal approximation may be less accurate for severe skew or heavy tails.

9. Let X_1 and X_2 be independent observations from a normal population with mean 10 and variance 4. Compare $A = X_1 + X_2$ and $B = 2X_1$. [5 marks]

(a) Find both distributions. [2 marks]

A adds two independent variances; B scales one variance by four.

Answer: $A \sim N(20, 8)$ and $B \sim N(20, 16)$.

(b) Which has the larger probability of exceeding 24? Give both probabilities. [3 marks]

Both have mean 20, but B has larger spread.

Answer: A: 0.0786; B: 0.1587. B is larger.

10. A report rounds a calculated normal tail probability of 0.00047 to two decimal places and reports zero. [4 marks]

(a) Explain why saying the event is impossible is not justified. [2 marks]

The unrounded model gives a positive area.

Answer: Rounding a small positive probability to zero does not make it impossible.

(b) Give a clearer probability report and one modelling qualification. [2 marks]

Tail probabilities can be sensitive to model assumptions and parameter estimates.

Answer: Report 0.00047 (about 0.047%), and state that the result depends on the normal model being appropriate.