Marking guide | Continuous probability and normal distributions

Practice test 5 - modelling and interpretation

10 questions | 43 marks | 75 minutes

Work with a two-piece density, compare alternative normal decisions, and evaluate modelling assumptions.

Equivalent correct methods earn the corresponding marks.

1. A density equals kx on $[0, 1]$ and $k(2 - x)$ on $(1, 2]$, and is zero outside $[0, 2]$. [4 marks]

(a) Find k. [2 marks]

The two triangles each have area $\frac{k}{2}$, so total area is k.

Answer: $k = 1$.

(b) Find $P(0.5 < X < 1.5)$. [2 marks]

The excluded tails are two triangles of base 0.5 and height 0.5, each area $\frac{1}{8}$.

Answer: $\frac{3}{4}$.

2. Use the two-piece density from Question 1. [5 marks]

(a) Give its CDF for x between 0 and 1, then for x between 1 and 2. [3 marks]

First integrate t from 0 to x.

After $x = 1$, start from accumulated area $\frac{1}{2}$ and integrate $2 - t$ from 1 to x.

Answer: $\frac{x^2}{2}$ then $2x - \frac{x^2}{2} - 1$.

(b) Find $P(X > 1.5 \mid X > 1)$. [2 marks]

The upper small triangle has area $\frac{1}{8}$; divide by the upper-half area $\frac{1}{2}$.

Answer: $\frac{1}{4}$.

3. The same distribution is symmetric around $x = 1$. [5 marks]

(a) Find $\mathrm{E}(X)$ by symmetry, then show $\mathrm{E}(X^2) = \frac{7}{6}$. [3 marks]

$\int_0^1 x^3\,dx + \int_1^2 x^2(2-x)\,dx = \frac{1}{4} + \frac{11}{12} = \frac{7}{6}$.

Answer: $\mathrm{E}(X) = 1$ and $\mathrm{E}(X^2) = \frac{7}{6}$.

(b) Find its variance and standard deviation. [2 marks]

Variance $= \frac{7}{6} - 1^2 = \frac{1}{6}$.

Answer: $\frac{1}{6}$ and $\frac{1}{\sqrt{6}}$.

4. A normally distributed dimension has mean 30 mm and SD 0.2 mm. Parts are accepted when between 29.7 and 30.4 mm. [5 marks]

(a) Find the probability of acceptance. [2 marks]

Standardise the acceptance limits to -1.5 and 2.

Answer: 0.9104.

(b) Which rejection tail is larger? Quantify both. [3 marks]

Calculate both tails rather than assuming symmetry about unequal limits.

Answer: Lower 0.0668; upper 0.0228; the lower tail is larger.

5. A normally distributed demand has mean 200 and SD 30. A business wants stock sufficient on 98% of days. [4 marks]

(a) Find the required model cut-off. [2 marks]

Use the 98th percentile.

Answer: 261.6125.

(b) If stock must be a whole number, what is the smallest sufficient stock under this model? [2 marks]

Round the model cut-off upward, since rounding down would fall below the target percentile.

Answer: 262.

6. Independent normal variables satisfy $A \sim N(12, 4)$, $B \sim N(8, 1)$ and $C \sim N(10, 9)$. [4 marks]

(a) Find the distribution of $A + B - C$. [2 marks]

Add the signed means; all independent variances add.

Answer: $N(10, 14)$.

(b) Find $P(A + B > C)$. [2 marks]

The event is $A + B - C > 0$. Standardise zero and take the upper tail.

Answer: 0.9962

7. Normal packet masses have mean 100 g and SD 5 g. A random independent sample contains 100 packets. [4 marks]

(a) Find the probability that the sample mean is below 99 g. [2 marks]

$\mathrm{SE} = 0.5$ g; the z-bound is -2.

Answer: 0.0228

(b) Would the probability that one packet is below 99 g be as small? Calculate it. [2 marks]

An individual uses SD 5 g and $z = -0.2$.

Answer: 0.4207

8. A population is strongly right-skewed. Its mean is 6 and SD is 4. [4 marks]

(a) For a random independent sample of 400, give the approximate mean distribution. [2 marks]

$\mathrm{SE} = \frac{4}{20} = 0.2$, so its variance is 0.04. State the CLT approximation.

Answer: $N(6, 0.04)$.

(b) Does this make the original population normal? [2 marks]

It is the distribution of the sample mean that is approximately normal.

Answer: No.

9. A factory combines equal numbers of outputs from machines with means 10 and 30. Each machine output is normal with SD 1. [4 marks]

(a) Explain why the combined individual-output distribution need not be normal. [2 marks]

A mixture of normal populations is different from adding independent normal values.

Answer: It can have two separated peaks near 10 and 30.

(b) What is the combined population mean? [2 marks]

Weight the two means equally; this does not require a normal combined shape.

Answer: 20.

10. A quality-control report says: “The model probability of exceeding the limit is 0.01, so exactly one of the next 100 outputs will exceed it.” [4 marks]

(a) Explain the error. [2 marks]

Random future outcomes fluctuate; a probability does not fix the realised total.

Answer: One is the expected count, not a guaranteed count.

(b) State the expected number exceeding the limit in 250 outputs. [2 marks]

Multiply the number of outputs by the exceedance probability. An expectation need not be an integer.

Answer: 2.5.