Marking guide | Continuous probability and normal distributions

Practice test 4 - reverse calculations and comparisons

10 questions | 42 marks | 65 minutes

Develop density and CDF reasoning, infer normal parameters, and distinguish correlated readings from independent samples.

Equivalent correct methods earn the corresponding marks.

1. A continuous model has density $f(x) = kx^2$ on $[0, 3]$. [4 marks]

(a) Find k. [2 marks]

$\int_0^3 x^2\,dx = 9$, so $9k = 1$.

Answer: $\frac{1}{9}$.

(b) Find $P(X > 2)$. [2 marks]

The CDF inside the support is $\frac{x^3}{27}$, so $P(X > 2) = 1 - \frac{8}{27}$.

Answer: $\frac{19}{27}$.

2. For the model $f(x) = \frac{x^2}{9}$ on $[0, 3]$, [4 marks]

(a) find the 80th percentile. [2 marks]

Solve $\frac{q^3}{27} = 0.8$ in the support.

Answer: $3\sqrt[3]{0.8} \approx 2.7850$.

(b) find $P(X > 2 \mid X > 1)$. [2 marks]

The numerator is $\frac{19}{27}$ and denominator $\frac{26}{27}$; divide.

Answer: $\frac{19}{26}$.

3. For density $\frac{x^2}{9}$ on $[0, 3]$, [5 marks]

(a) find the first and second moments. [3 marks]

Integrate $\frac{x^3}{9}$ and $\frac{x^4}{9}$ over $[0, 3]$.

Answer: $\mathrm{E}(X) = \frac{9}{4}$, $\mathrm{E}(X^2) = \frac{27}{5}$.

(b) find the variance and the variance of $Y = 2X - 1$. [2 marks]

$\frac{27}{5} - (\frac{9}{4})^2 = \frac{27}{80}$; multiply by 2^2.

Answer: $\frac{27}{80}$ and $\frac{27}{20}$.

4. A machine output is modelled as normal. Its 5th and 95th percentiles are 28 and 44. [5 marks]

(a) Find its mean. [2 marks]

The quantiles are symmetric around the mean.

Answer: 36.

(b) Find its standard deviation. [3 marks]

$44 = 36 + \Phi^{-1}(0.95)\sigma$.

Answer: 4.8637.

5. $X \sim N(60, 25)$. [4 marks]

(a) Find $P(X < 55 \text{ or } X > 70)$. [2 marks]

The two tails are disjoint; add the areas for $z < -1$ and $z > 2$.

Answer: 0.1814.

(b) Find the cut-off c such that $P(X > c) = 0.025$. [2 marks]

Use the 97.5th percentile.

Answer: 69.7998.

6. A normal test score has mean 100 and SD 15. Two students score 118 and 85. [4 marks]

(a) Find both z-scores. [2 marks]

Subtract 100 and divide by 15.

Answer: 1.2 and -1.

(b) Find the proportion of scores between the two students. [2 marks]

Subtract the lower normal cumulative probability from the upper.

Answer: 0.7263.

7. Independent normal X and Y have means 8 and 3, and variances 1 and 4. [4 marks]

(a) Find the mean and variance of $W = 3X - 2Y$. [2 marks]

Mean $3(8) - 2(3) = 18$; variance $9(1) + 4(4) = 25$.

Answer: 18 and 25.

(b) Find $P(W < 23)$. [2 marks]

Use mean 18 and SD 5.

Answer: 0.8413

8. A normal population has mean 250 and SD 20. Independent samples of 16 observations are taken. [4 marks]

(a) Find the probability that a sample mean lies between 245 and 260. [2 marks]

$\mathrm{SE} = 5$; the z-bounds are -1 and 2.

Answer: 0.8186.

(b) Find the probability that a total of 16 observations exceeds 4160. [2 marks]

The total has mean 4000 and SD 80; its z-bound is 2.

Answer: 0.0228

9. A lab records the same object ten times with an instrument having a shared unknown calibration error. [4 marks]

(a) Why might the ten readings fail an independence assumption? [2 marks]

The joint dependence is not removed by simply repeating the reading.

Answer: A shared calibration error can shift all readings together.

(b) Would dividing a claimed population SD by the square root of ten automatically account for that calibration error? [2 marks]

The usual SE formula assumes appropriately independent errors and does not remove systematic error.

Answer: No.

10. A density curve has 40% of its area below 2 and 75% below 5. [4 marks]

Cumulative probability information

x	$F(x)$
2	0.40
5	0.75

(a) Find $P(2 < X < 5)$. [2 marks]

Subtract 0.40 from 0.75.

Answer: 0.35.

(b) Find $P(X < 5 \mid X > 2)$. [2 marks]

The numerator is 0.35 and denominator 0.60, giving $\frac{0.35}{0.60} = \frac{7}{12}$.

Answer: $\frac{7}{12}$.