Practice test 2 - cumulative models and cut-offs

10 questions | 42 marks | 65 minutes

Use changing density shapes, CDFs, reverse normal calculations and weighted measurements.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = k(2 - x)$ on $[0, 2]$, zero elsewhere. [4 marks]

(a) Find k. [2 marks]

$\int_0^2 (2 - x)\,dx = 2$, so $2k = 1$.

Answer: $\frac{1}{2}$.

(b) Find $P(X < 1)$. [2 marks]

$\int_0^1 (1 - \frac{x}{2})\,dx = 1 - \frac{1}{4} = \frac{3}{4}$.

Answer: $\frac{3}{4}$.

2. For density $1 - \frac{x}{2}$ on $[0, 2]$, [5 marks]

(a) find the CDF on the whole real line. [3 marks]

Integrate the density from 0 to x, then add outer cumulative branches.

Answer: 0 below 0; $x - \frac{x^2}{4}$ on $[0, 2]$; 1 above 2.

(b) find the median. [2 marks]

Solve $m - \frac{m^2}{4} = \frac{1}{2}$; of the two quadratic roots only $2 - \sqrt{2}$ lies in $[0, 2]$.

Answer: $2 - \sqrt{2}$.

3. A density equals c on $[0, 1]$ and $2c$ on $(1, 2]$, and is zero elsewhere. [4 marks]

(a) Find c and $P(X < 0.5)$. [2 marks]

Total area $c + 2c = 1$. The first half-unit has area $\frac{c}{2}$.

Answer: $c = \frac{1}{3}$; probability $\frac{1}{6}$.

(b) Does the CDF jump at $x = 1$? Explain. [2 marks]

The density height jumps, but the cumulative area is continuous and the point has probability zero.

Answer: No.

4. U is uniform on $[-2, 4]$. [5 marks]

(a) Find $\mathrm{E}(U)$, $\mathrm{Var}(U)$ and $\mathrm{SD}(U)$. [3 marks]

Mean is the midpoint; variance is $\frac{(4-(-2))^2}{12} = 3$.

Answer: 1, 3 and $\sqrt{3}$.

(b) Find the mean and variance of $V = 3U + 2$. [2 marks]

$\mathrm{E}(V) = 3(1) + 2$; $\mathrm{Var}(V) = 9(3)$.

Answer: 5 and 27.

5. $X \sim N(24, 2.25)$ models filling time in seconds. [4 marks]

(a) Find $P(X > 27)$. [2 marks]

SD is 1.5; the z-score is 2.

Answer: 0.0228

(b) Find the central 80% interval. [2 marks]

Each tail has area 0.10; use the 10th and 90th percentiles.

Answer: $[22.0777, 25.9223]$ seconds.

6. A normal model has known mean 100. Its 90th percentile is 112. [4 marks]

(a) Find its standard deviation. [2 marks]

$112 = 100 + \Phi^{-1}(0.90)\sigma$; solve for positive σ.

Answer: 9.3636.

(b) State its 10th percentile. [2 marks]

The 10th and 90th percentiles are symmetric about mean 100.

Answer: 88.

7. Independent $X \sim N(10, 4)$ and $Y \sim N(5, 9)$. [4 marks]

(a) Find the distribution of $2X - Y$. [2 marks]

Mean $2(10) - 5 = 15$; variance $4(4) + 9 = 25$.

Answer: $N(15, 25)$.

(b) Find $P(2X - Y < 10)$. [2 marks]

The z-score is $\frac{10 - 15}{5} = -1$.

Answer: 0.1587

8. A normal population has mean 60 and SD 10. An independent random sample of 25 values is selected. [4 marks]

(a) Find the SE and $P(\text{sample mean} > 64)$. [2 marks]

$\frac{10}{\sqrt{25}} = 2$; standardised bound is 2.

Answer: SE 2; probability 0.0228.

(b) Would $P(\text{an individual value} > 64)$ be the same? Calculate it. [2 marks]

For an individual use SD 10, giving $z = 0.4$, not 2.

Answer: 0.3446

9. A non-normal population has mean 18 and finite SD 6. [4 marks]

(a) For $n = 100$ independent random observations, give an approximate distribution of the sample mean. [2 marks]

SE is 0.6; the variance is 0.36.

Answer: $N(18, 0.36)$, if the CLT approximation is adequate.

(b) Identify one feature that could make convergence to normality slower. [2 marks]

The population shape affects how large a sample is needed.

Answer: Strong skew or a heavy-tailed population with rare extreme values.

10. A sensor rounds continuous measurements to the nearest whole number. The underlying X has density $2x$ on $[0, 1]$. [4 marks]

(a) Find $P(\text{the displayed value is } 1)$. [2 marks]

This corresponds to $X \geq 0.5$ apart from an endpoint, so area is $1 - 0.5^2 = \frac{3}{4}$.

Answer: $\frac{3}{4}$.

(b) Explain why the displayed value can have positive probability although $P(X = 1) = 0$. [2 marks]

The underlying continuous singleton and the rounded event are different events.

Answer: Rounding combines an entire interval into one display category.