Marking guide | Continuous probability and normal distributions

Practice test 1 - areas and normal models

10 questions | 41 marks | 65 minutes

Build from valid densities and interval areas to normal cut-offs, independent totals and sample means.

Equivalent correct methods earn the corresponding marks.

1. A model has density $f(x) = kx$ for $0 \le x \le 2$, and zero elsewhere. [4 marks]

(a) Find k and check that this gives a valid density. [2 marks]

$\int_0^2 kx\,dx = 2k = 1$. The resulting density is nonnegative.

Answer: $k = \frac{1}{2}$.

(b) Find $P(0.5 < X < 1.5)$. [2 marks]

$\int_{0.5}^{1.5} \frac{x}{2}\,dx = [\frac{x^2}{4}]_{0.5}^{1.5} = \frac{1}{2}$.

Answer: $\frac{1}{2}$.

2. A bus waiting time W is uniform on $[1, 7]$ minutes. [4 marks]

(a) State the density and find $P(W < 3)$. [2 marks]

The support width is 6; the event width is 2.

Answer: Density $\frac{1}{6}$ on $[1, 7]$; probability $\frac{1}{3}$.

(b) Given that $W > 3$, find the chance $W > 5$. [2 marks]

Conditional widths are 2 out of 4; equivalently divide areas $\frac{2}{6}$ by $\frac{4}{6}$.

Answer: $\frac{1}{2}$.

3. A continuous CDF equals x^3 on $[0,1]$, with outer values zero and one. [4 marks]

(a) Find the density and $P(0.2 < X < 0.8)$. [2 marks]

Differentiate the interior CDF; $0.8^3 - 0.2^3 = 0.504$.

Answer: $3x^2$; 0.504.

(b) Find the median. [2 marks]

Solve $m^3 = 0.5$ in $[0,1]$.

Answer: $\sqrt[3]{\frac{1}{2}} \approx 0.7937$.

4. X has density $2x$ on $[0,1]$. [5 marks]

(a) Find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$. [3 marks]

Integrate $2x^2$ and $2x^3$ from zero to one.

Answer: $\frac{2}{3}$ and $\frac{1}{2}$.

(b) Find $\mathrm{Var}(X)$ and the standard deviation. [2 marks]

$\frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}$; take its positive square root.

Answer: $\frac{1}{18}$ and $\frac{1}{\sqrt{18}}$.

5. $Y \sim N(50, 16)$, using mean-variance notation. [4 marks]

(a) State the standard deviation and standardise $Y = 56$. [2 marks]

$\sqrt{16} = 4$ and $\frac{56-50}{4} = 1.5$.

Answer: SD 4; $z = 1.5$.

(b) Find $P(46 < Y < 56)$ to four decimals. [2 marks]

The z-bounds are -1 and 1.5; subtract their cumulative areas.

Answer: 0.7745.

6. A normal component length has mean 80 mm and SD 3 mm. [4 marks]

(a) Find the length exceeded by 10% of components. [2 marks]

Use the 90th percentile, $80 + 3\Phi^{-1}(0.90)$.

Answer: 83.8447 mm.

(b) Explain why the 10th percentile would answer a different question. [2 marks]

The cumulative area of 0.10 describes the lowest tenth.

Answer: It leaves 10% below the cut-off, not above.

7. Independent $A \sim N(20, 9)$ and $B \sim N(12, 4)$ model two processing times in minutes. [4 marks]

(a) Find the distribution of $T = A + B$. [2 marks]

Add means and independent variances.

Answer: $T \sim N(32, 13)$.

(b) Find $P(T > 36)$. [2 marks]

Standardise with mean 32 and SD $\sqrt{13}$, then use an upper tail.

Answer: 0.1336

8. Independent packet masses come from a normal population with mean 500 g and SD 12 g. A sample contains 9 packets. [4 marks]

(a) Find the distribution of the sample mean. [2 marks]

The variance is $\frac{144}{9} = 16$.

Answer: $N(500, 16)$.

(b) Find $P(\text{sample mean} < 496)$. [2 marks]

SE is 4 g, so the standardised bound is -1.

Answer: 0.1587

9. A student models two times as normal and then adds their variances. [4 marks]

(a) What further assumption is needed? [2 marks]

Normal marginal distributions do not guarantee independence.

Answer: The two times must be independent, or at least have zero covariance for the variance calculation.

(b) If A and B are the same random reading X with $\mathrm{Var}(X) = 4$, compare $\mathrm{Var}(A + B)$ with the claimed 8. [2 marks]

$A + B = 2X$, so its variance is $4\mathrm{Var}(X) = 16$.

Answer: The actual variance is 16.

10. A company estimates the mean life of a product using a large sample of volunteers who report a failed product. [4 marks]

(a) Explain why the large sample may still be biased. [2 marks]

More observations do not remove this selection mechanism.

Answer: People with failures are selectively included; the sample need not represent all products.

(b) State one change that would support a valid sample-mean model. [2 marks]

Random selection and appropriately independent observations support the stated sampling assumptions.

Answer: Randomly select products from the full production population and follow them consistently.