Marking guide | Discrete probability distributions

Discrete distributions - Applications and reasoning test

10 questions | 72 marks | 75 minutes

An integrated paper requiring distribution construction, nonlinear costs, model criticism and justified decisions.

Equivalent correct methods earn the corresponding marks.

1. A count X takes values 0, 1 and 3 with probabilities a, b and 0.2. Its expected value is 1.1. [7 marks]

(a) Find a and b. [3 marks]

Expectation gives $b + 3(0.2) = 1.1$, hence $b = 0.5$.

Total probability gives $a = 1 - 0.5 - 0.2$.

Answer: $a = 0.3$, $b = 0.5$.

(b) Find $\mathrm{Var}(X)$. [2 marks]

$E(X^2) = 0.5 + 9(0.2) = 2.3$.

Variance $= 2.3 - 1.1^2$.

Answer: 1.09.

(c) Find $P(X \leq 1)$ and state whether observing 3 is impossible. [2 marks]

Add $a + b$ for the cumulative event.

Answer: 0.8; observing 3 remains possible with probability 0.2.

2. An integer-valued X has $F(0)=a$, $F(1)=3a$, $F(2)=1-a$ and $F(3)=1$. No other values occur. Also $P(X=1)=0.4$. [7 marks]

(a) Find a and the probability table. [3 marks]

The jump at 1 is $3a-a=0.4$, so $a=0.2$.

Successive jumps are 0.2, 0.4, 0.2 and 0.2.

Answer: $a=0.2$; probabilities 0.2, 0.4, 0.2, 0.2 at 0, 1, 2, 3.

(b) Find the mean and variance. [2 marks]

$E(X)=0.4+0.4+0.6=1.4$.

$E(X^2)=0.4+0.8+1.8=3$; variance $=3-1.96$.

Answer: Mean 1.4; variance 1.04.

(c) Find $P(1<X\leq 3)$. [2 marks]

Use $1-F(1)=1-0.6$.

Answer: 0.4.

3. A game prize X is 0 dollars, 3 dollars or 12 dollars with probabilities 0.6, 0.3 and 0.1. [7 marks]

(a) Find its fair entry fee and prize variance. [3 marks]

$E(X)=3(0.3)+12(0.1)=2.1$.

$E(X^2)=9(0.3)+144(0.1)=17.1$; variance $=17.1-2.1^2$.

Answer: Fair fee 2.10 dollars; variance 12.69 dollars².

(b) A promotion gives twice the prize but charges 5 dollars. Find expected net gain and its variance. [2 marks]

$Y=2X-5$.

$E(Y)=4.2-5$; $\mathrm{Var}(Y)=4(12.69)$.

Answer: Mean −0.80 dollars; variance 50.76 dollars².

(c) Explain why doubling prizes does not necessarily benefit the player on average. [2 marks]

Here the fee 5 exceeds the new expected gross prize 4.2.

Answer: The increased fee may exceed the increase in expected prize.

4. A policy has a claim of 1000 dollars with probability 0.02 and no claim otherwise. Administration costs 8 dollars per policy. The premium c must be a whole number of dollars. [8 marks]

(a) Find the expected claim and the smallest premium giving expected profit at least 15 dollars. [3 marks]

Expected claim $= 1000(0.02) = 20$.

Expected profit $= c - 8 - 20 \geq 15$, giving $c \geq 43$.

Answer: Expected claim 20 dollars; minimum premium 43 dollars.

(b) At that premium, find the probability of a negative profit on a single policy. [2 marks]

Without a claim profit is $43 - 8 = 35$; with a claim it is -965.

Only the claim event produces a loss.

Answer: 0.02.

(c) Find the standard deviation of single-policy profit. [3 marks]

Claim variance $= 1000^2(0.02)(0.98) = 19600$.

Subtracting a claim and adding fixed charges preserves its SD: $\sqrt{19600} = 140$.

Answer: 140 dollars.

5. A usage count X is 0, 1 or 2 with probabilities 0.5, 0.3 and 0.2. A nonlinear cost is $C = 5 + 2X + X^2$ dollars. [7 marks]

(a) Construct the distribution of C. [2 marks]

Evaluate the cost separately at $x = 0$, 1 and 2.

Answer: $C = 5, 8, 13$ with probabilities 0.5, 0.3, 0.2.

(b) Find $E(C)$ and $\mathrm{Var}(C)$. [3 marks]

$E(C) = 5(0.5) + 8(0.3) + 13(0.2) = 7.5.$

$E(C^2) = 25(0.5) + 64(0.3) + 169(0.2) = 65.5.$

Variance $= 65.5 - 7.5^2.$

Answer: Mean 7.50 dollars; variance 9.25 dollars².

(c) Why should the affine variance rule not be applied directly to the whole expression for C ? [2 marks]

Construct the transformed distribution or calculate the necessary higher moments.

Answer: The X^2 term makes the transformation nonlinear.

6. Independent daily demands are $A = 0$ or 1 with probabilities 0.4 and 0.6, and $B = 0$ or 2 with probabilities 0.7 and 0.3. Set $T = A + B$. [7 marks]

(a) Find the distribution of T. [3 marks]

Multiply probabilities of the four independent pairs: $(0, 0)$, $(1, 0)$, $(0, 2)$, $(1, 2)$.

Answer: $P(T = 0, 1, 2, 3) = 0.28, 0.42, 0.12, 0.18.$

(b) Find $E(T)$ and $\mathrm{Var}(T)$. [2 marks]

A has mean 0.6 and variance 0.24.

B has mean 0.6 and variance $4(0.3)(0.7) = 0.84$; add under independence.

Answer: Mean 1.2; variance 1.08.

(c) Would the probability table in (a) still follow from the marginal tables if independence were removed? [2 marks]

The products used for the joint pairs would no longer be justified.

Answer: No.

7. Use the distribution $T = 0, 1, 2, 3$ with probabilities 0.28, 0.42, 0.12, 0.18. A shortage occurs when $T \geq 2$. Assume seven independent days follow this same distribution. [6 marks]

(a) Define a shortage indicator and give its mean and variance. [2 marks]

Shortage probability $= 0.12 + 0.18 = 0.3$.

A Bernoulli indicator has variance $0.3(0.7)$.

Answer: $I = 1$ for $T \geq 2$, otherwise 0; mean 0.3 and variance 0.21.

(b) State the distribution, mean and variance of the number S of shortage days. [2 marks]

Seven fixed independent days have common shortage probability 0.3.

Use np and $np(1-p)$.

Answer: $S \sim \text{Bin}(7, 0.3)$; mean 2.1, variance 1.47.

(c) Find $P(S \geq 3)$. [2 marks]

Use $1 - P(S \leq 2)$.

$\sum_{k=3}^{7} \binom{7}{k}(0.3)^k(0.7)^{7-k} = 0.3529$.

Answer: 0.3529

8. Each batch has one shared environment: equally likely low-risk or high-risk. Conditional on that environment, two items fail independently with probabilities 0.2 or 0.8 respectively. X counts failed items. [8 marks]

(a) Find each item's unconditional failure probability. [2 marks]

$0.5(0.2) + 0.5(0.8) = 0.5.$

Answer: 0.5.

(b) Construct the unconditional distribution of X. [3 marks]

$P(X = 2) = 0.5(0.2^2) + 0.5(0.8^2) = 0.34.$

By symmetry $P(X = 0) = 0.34$; the remaining probability is 0.32.

Answer: $P(X = 0) = 0.34$; $P(X = 1) = 0.32$; $P(X = 2) = 0.34$.

(c) Compare its variance with $\text{Bin}(2, 0.5)$ and identify the failed assumption. [3 marks]

$E(X) = 1$; $E(X^2) = 0.32 + 4(0.34) = 1.68.$

The shared environment links the outcomes even though their marginal probabilities match.

Answer: Actual variance 0.68; binomial variance 0.5. Unconditional independence fails.

9. For independent trials with success probability 0.1, define X_n as the success count in n trials. A design requires $P(X_n \geq 5) \leq 0.05$. [8 marks]

(a) Evaluate the tail probability for $n = 19$, 20 and 21. [3 marks]

For each n use $\text{Bin}(n, 0.1)$ and calculate $1 - P(X_n \leq 4)$.

Answer: 19: 0.0352; 20: 0.0432; 21: 0.0522.

(b) Use the three values to conjecture how $P(X_n \geq 5)$ changes as n increases with p fixed. State the largest acceptable n among 19, 20 and 21. [2 marks]

The computed values suggest that $P(X_n \geq 5)$ is nondecreasing as n increases. [1 mark]

At $n = 20$ the probability is below 0.05, whereas at $n = 21$ it is above 0.05; choose $n = 20$. [1 mark]

Answer: Conjecture that the tail probability is nondecreasing in n; the largest acceptable value in the stated set is $n = 20$.

(c) Justify your conjecture using the relationship between counts in n and $n + 1$ trials, with p fixed. [3 marks]

Use the same first n independent trials in both experiments and let B be the success indicator of the next trial. Then $X_{n+1} = X_n + B$, with $B \in \{0, 1\}$. [1 mark]

Every outcome with $X_n \geq 5$ also has $X_{n+1} \geq 5$, since the added trial cannot remove a success. [1 mark]

Therefore $P(X_{n+1} \geq 5) \geq P(X_n \geq 5)$, proving the conjectured nondecreasing trend. [1 mark]

Answer: Adding extra trials cannot reduce an existing success count.

10. A binomial model for a daily count has exact mean 6 and variance 4.5. An alert is raised for a count of at least 10. [7 marks]

(a) Find n and p. [3 marks]

$1 - p = \frac{4.5}{6} = 0.75.$

$p = 0.25$ and $n = \frac{6}{0.25} = 24.$

Answer: $n = 24,\ p = 0.25.$

(b) Find the probability of an alert. [2 marks]

Use $X \sim \text{Bin}(24, 0.25)$ and $P(X \geq 10) = 1 - P(X \leq 9).$

$\sum_{k=10}^{24} \binom{24}{k}(0.25)^k(0.75)^{24-k} = 0.0547.$

Answer: 0.0547

(c) If the true success probability changes to 0.2, may the old alert probability be used unchanged? Explain. [2 marks]

The model parameters have changed.

Recalculate the $\text{Bin}(24, 0.2)$ tail; old mean/variance and alert probabilities no longer apply.

Answer: No.