Discrete distributions - Practice test 5

10 questions | 65 marks | 75 minutes

Connect cumulative thresholds, nonlinear transformations, costs and repeated sampling without losing the model assumptions.

Equivalent correct methods earn the corresponding marks.

1. A fair spinner has five equal regions labelled 0, 0, 1, 3 and 3. X is the label on one spin. [6 marks]

(a) Construct the probability distribution of X. [2 marks]

Equal regions, rather than distinct labels, are equally likely.

Answer: Probabilities 0.4, 0.2 and 0.4 at 0, 1 and 3.

(b) Find $E(X)$ and $\mathrm{Var}(X)$. [2 marks]

$E(X) = 0.2 + 3(0.4) = 1.4$.

$E(X^2) = 0.2 + 9(0.4) = 3.8$; variance $= 3.8 - 1.4^2$.

Answer: Mean 1.4; variance 1.84.

(c) Find $P(X \geq 1)$. [2 marks]

Add the probabilities at 1 and 3.

Answer: 0.6.

2. X takes 0, 1, 2 and 3 with probabilities 0.1, 0.25, 0.4 and 0.25. [6 marks]

(a) Give its cumulative probabilities at the four values. [2 marks]

Add the probabilities successively.

Answer: 0.1, 0.35, 0.75, 1.

(b) Find the smallest value x for which $P(X \leq x) \geq 0.5$. [2 marks]

$F(1) = 0.35$ is below 0.5, but $F(2) = 0.75$ is at least 0.5.

Answer: $x = 2$.

(c) Find the smallest x for which $P(X \leq x) \geq 0.9$. [2 marks]

$F(2) = 0.75$ is insufficient; $F(3) = 1$ meets the threshold.

Answer: $x = 3$.

3. A traveller chooses protection plan A costing a fixed 10 dollars, or plan B costing 4 dollars plus an excess of 50 dollars if a claim occurs. Claim probability is 0.1; there can be at most one claim. [6 marks]

(a) Find the expected cost of each plan. [2 marks]

B costs $4+50B_0$, where B_0 is a claim indicator with $p=0.1$.

$E(\text{cost } B)=4+50(0.1)=9$.

Answer: A: 10 dollars; B: 9 dollars.

(b) Find the variance of cost for each plan. [2 marks]

A is fixed.

$\text{Var}(\text{cost } B)=50^2(0.1)(0.9)=225$.

Answer: A: 0; B: 225 dollars².

(c) Explain why choosing A need not be irrational despite its higher expected cost. [2 marks]

A decision can value certainty or require a maximum affordable payment as well as a low mean.

Answer: It removes the possibility of the cost of 54 dollars under B.

4. X takes 0, 1 and 2 with probabilities a, $1-2a$ and a. [7 marks]

(a) Show that $E(X)=1$ for every valid a. [2 marks]

$0(a)+1(1-2a)+2a=1$.

Answer: $E(X)=1$.

(b) Given $\text{Var}(X)=0.4$, find a and all probabilities. [3 marks]

$E(X^2)=1-2a+4a=1+2a$.

Variance is $2a$; solve $2a=0.4$.

Answer: $a=0.2$; probabilities 0.2, 0.6, 0.2.

(c) Could the same family have variance 1.2? Explain. [2 marks]

That would require $a=0.6$, giving the negative middle probability -0.2.

Answer: No.

5. X is 0, 1 or 2 with probabilities 0.5, 0.25 and 0.25. Let $Y = X^2$. [7 marks]

(a) Find $E(X)$ and $\text{Var}(X)$. [2 marks]

$E(X) = 0.25 + 0.5 = 0.75$.

$E(X^2) = 0.25 + 1 = 1.25$; subtract 0.75^2.

Answer: Mean 0.75; variance 0.6875.

(b) Find $E(Y)$ and $\text{Var}(Y)$ directly. [3 marks]

Y has values 0, 1 and 4 with the original probabilities.

$E(Y^2) = 0.25 + 16(0.25) = 4.25$; variance $= 4.25 - 1.25^2$.

Answer: Mean 1.25; variance 2.6875.

(c) Why can the linear variance rule not be used with $a = 2$? [2 marks]

$Y = X^2$ is nonlinear, so use its actual distribution.

Answer: Squaring X is not the same transformation as multiplying X by 2.

6. Sixteen independent copies of X each have mean 5 and variance 4. Let T be their sum and A their average. [7 marks]

(a) Find $E(T)$ and $\text{Var}(T)$. [2 marks]

Add 16 means and 16 independent variances.

Answer: 80 and 64.

(b) Find $E(A)$, $\text{Var}(A)$ and $\text{SD}(A)$. [3 marks]

$A = \frac{T}{16}$, so its mean is $\frac{80}{16} = 5$.

Its variance is $\frac{64}{16^2} = 0.25$; SD is 0.5.

Answer: 5, 0.25 and 0.5.

(c) Explain why the average has smaller spread without a smaller mean. [2 marks]

The common location remains 5, while the variance becomes $\frac{\text{Var}(X)}{16}$.

Answer: Averaging independent observations reduces variability around the common mean.

7. The joint probabilities of indicators B and C are shown. [7 marks]

Joint probabilities

B	C	Probability
0	0	0.4
1	0	0.2
0	1	0.1
1	1	0.3

(a) Find $P(B=1)$ and $P(C=1)$. [2 marks]

Sum the relevant rows: $B=1$ gives $0.2+0.3$; $C=1$ gives $0.1+0.3$.

Answer: 0.5 and 0.4.

(b) Find the distribution and variance of $T=B+C$. [3 marks]

$E(T)=0.3+0.6=0.9$.

$E(T^2)=0.3+1.2=1.5$; variance $=1.5-0.9^2$.

Answer: $P(T=0)=0.4$, $P(T=1)=0.3$, $P(T=2)=0.3$; variance 0.69.

(c) Test independence and compare with adding their separate variances. [2 marks]

$P(B=1, C=1)=0.3$ differs from $0.5(0.4)=0.2$.

Individual variances are 0.25 and 0.24.

Answer: They are dependent; adding separate variances gives 0.49, not 0.69.

8. In 20 tosses of a new coin, 10 heads and 10 tails are observed. [6 marks]

(a) Give the relative-frequency estimate of its head probability. [2 marks]

$\frac{10}{20} = 0.5$.

Answer: 0.5.

(b) Does this result prove future tosses are independent with $p = 0.5$? Explain. [2 marks]

An observed proportion estimates p but does not prove it.

A tally alone also does not establish independence.

Answer: No.

(c) State a conditional binomial model for heads in the next 12 tosses if those assumptions are adopted. [2 marks]

The model is conditional on a stable $p = 0.5$ and independent future tosses.

Answer: $X \sim \text{Bin}(12, 0.5)$.

9. $X \sim \text{Bin}(25, 0.12)$. [6 marks]

(a) Find $P(2 \le X \le 5)$, to four decimal places. [2 marks]

Use $P(X \le 5) - P(X \le 1)$.

$\sum_{k=2}^{5} \binom{25}{k} (0.12)^k (0.88)^{25-k} = 0.7486$.

Answer: 0.7486

(b) Find $P(X \ge 6)$. [2 marks]

Use $1 - P(X \le 5)$.

$\sum_{k=6}^{25} \binom{25}{k} (0.12)^k (0.88)^{25-k} = 0.0709$.

Answer: 0.0709

(c) Out of 100 comparable samples, how many are expected to have at least six successes? [2 marks]

Multiply the unrounded probability in (b) by 100.

Answer: 7.0881

10. An independent trial succeeds with probability 0.05. A team needs at least one success with probability at least 0.95. [7 marks]

(a) Form an inequality for the required trial count n. [2 marks]

Use the complement of n failures.

Answer: $1-0.95^{n}\geq 0.95$.

(b) Find the smallest n and verify the adjacent boundary. [3 marks]

$n\geq \frac{\ln(0.05)}{\ln(0.95)}\approx 58.404$.

$n=58$ gives 0.9490; $n=59$ gives 0.9515.

Answer: 59.

(c) Would knowing that the mean number of successes is above 1 be enough to guarantee the target? [2 marks]

The target concerns a specific tail probability, not merely the mean np.

Answer: No.