Discrete distributions - Practice test 4

10 questions | 64 marks | 70 minutes

Explain distribution identities, distinguish dependent indicators and solve a minimum-trial problem.

Equivalent correct methods earn the corresponding marks.

1. X takes values 0, 1, 2 and 3, with $P(X=x)=k(x+1)$. [6 marks]

(a) Find k and $P(X\geq 2)$. [2 marks]

The weights sum to $1+2+3+4=10$, so $k=0.1$.

The final two weights are 3 and 4.

Answer: $k=0.1$; $P(X\geq 2)=0.7$.

(b) Find $E(X)$ and $\mathrm{Var}(X)$. [2 marks]

$E(X)=0.2+0.6+1.2=2$.

$E(X^2)=0.2+1.2+3.6=5$; variance $=5-4$.

Answer: Mean 2; variance 1.

(c) Find $F(1.5)$. [2 marks]

Only 0 and 1 are no larger than 1.5; add $0.1+0.2$.

Answer: 0.3.

2. X takes values 0, 1, 2 and 3. Its cumulative values are $F(0)=0.15$, $F(1)=a$, $F(2)=0.8$ and $F(3)=1$. Also $P(X=1)=0.35$. [6 marks]

(a) Find a. [2 marks]

The jump at 1 is $a-0.15=0.35$.

Answer: $a=0.5$.

(b) Find $P(X=2)$. [2 marks]

$F(2)-F(1)=0.8-0.5$.

Answer: 0.3.

(c) Find $P(X\geq 2)$ and explain the endpoint. [2 marks]

Use $1-F(1)$, since the value 2 must remain included.

Answer: 0.5.

3. X takes -2, 0 and 2 with probabilities $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$. Let $Y=X^2$. [7 marks]

(a) Find $E(X)$, $E(X^2)$ and $\text{Var}(X)$. [2 marks]

The symmetric distribution has mean 0.

$E(X^2)=4(\frac{1}{4})+4(\frac{1}{4})=2$; variance is $2-0$.

Answer: 0, 2 and 2.

(b) Construct the distribution of Y and find its variance. [3 marks]

Combine $X=-2$ and $X=2$ at the same transformed value 4.

$E(Y)=2$; $E(Y^2)=16(\frac{1}{2})=8$; variance $=8-4$.

Answer: $P(Y=0)=\frac{1}{2}$, $P(Y=4)=\frac{1}{2}$; $\text{Var}(Y)=4$.

(c) Explain why $E(X^2)$ is not $[E(X)]^2$ here. [2 marks]

Expectation does not generally commute with a nonlinear operation.

Answer: They are 2 and 0, respectively.

4. D is the score on a fair six-sided die. [6 marks]

(a) Calculate $E(D^2)$. [2 marks]

$E(D^2) = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$.

Answer: $\frac{91}{6}$.

(b) Use $E(D) = 3.5$ to find $\mathrm{Var}(D)$. [2 marks]

$\frac{91}{6} - (\frac{7}{2})^2 = \frac{182}{12} - \frac{147}{12} = \frac{35}{12}$.

Answer: $\frac{35}{12}$.

(c) For two independent dice, find the variance of their total. [2 marks]

Add the two independent variances: $2(\frac{35}{12})$.

Answer: $\frac{35}{6}$.

5. X has mean $\mu = 30$ and standard deviation $\sigma = 6$. Define $Z = \frac{X-30}{6}$. [6 marks]

(a) Find $E(Z)$ and $\mathrm{Var}(Z)$. [2 marks]

Mean $= \frac{30-30}{6} = 0$.

Variance $= \frac{36}{6^2} = 1$.

Answer: 0 and 1.

(b) Express the event $Z > 2$ as an event involving X. [2 marks]

Multiply by positive 6 and add 30: $X - 30 > 12$.

Answer: $X > 42$.

(c) Can $P(Z > 2)$ be obtained from the supplied mean and SD alone? [2 marks]

The full distribution has not been specified; standardising does not automatically make it normal.

Answer: No.

6. Three independent indicators have success probabilities 0.1, 0.4 and 0.8. T is their total. [6 marks]

(a) Find its mean and variance. [2 marks]

Add the three means.

Add $p(1-p)$: $0.09+0.24+0.16=0.49$.

Answer: Mean 1.3; variance 0.49.

(b) Find $P(T=0)$ and $P(T=3)$. [2 marks]

No success: $0.9(0.6)(0.2)$.

All succeed: $0.1(0.4)(0.8)$.

Answer: 0.108 and 0.032.

(c) Explain why matching the mean with $\text{Bin}(3, \frac{1.3}{3})$ does not establish that model. [2 marks]

For example, the binomial variance would be $1.3(1-\frac{1.3}{3})$, not 0.49.

Answer: The trial probabilities are not equal.

7. A fair coin defines $B=1$ for heads. Let $C=1-B$ and $T=B+C$. [6 marks]

(a) Find $E(T)$ and $\text{Var}(T)$ directly. [2 marks]

T is identically 1 whatever the coin outcome.

Answer: Mean 1; variance 0.

(b) Calculate what an incorrect independence assumption would give for $\text{Var}(T)$. [2 marks]

Each individual indicator has variance $\frac{1}{4}$.

Adding them as if independent would give $\frac{1}{2}$.

Answer: 0.5.

(c) Explain the dependence. [2 marks]

If $B=1$ then $C=0$, and conversely.

Answer: Knowing B fixes C exactly.

8. $X \sim \text{Bin}(2, p)$, where $0 < p < 1$, and $P(X = 2) = 9P(X = 0)$. [7 marks]

(a) Form and solve an equation for p. [3 marks]

$p^2 = 9(1-p)^2$.

Both p and $1-p$ are positive, so $p = 3(1-p)$, giving $p = \frac{3}{4}$.

Answer: $p = \frac{3}{4}$.

(b) Find $P(X = 1)$. [2 marks]

$2p(1-p) = 2(\frac{3}{4})(\frac{1}{4})$.

Answer: $\frac{3}{8}$.

(c) Find the mean and variance. [2 marks]

$np = 2(\frac{3}{4})$; $np(1-p) = 2(\frac{3}{4})(\frac{1}{4})$.

Answer: Mean $\frac{3}{2}$; variance $\frac{3}{8}$.

9. Independent trials have success probability 0.2. A planner wants the probability of at least one success to be at least 0.95. [7 marks]

(a) Write the required inequality in n. [2 marks]

Use the complement of zero successes.

Answer: $1 - 0.8^n \geq 0.95$.

(b) Find the smallest integer n. [3 marks]

The inequality is $0.8^n \leq 0.05$.

$n \geq \frac{\ln(0.05)}{\ln(0.8)} \approx 13.425$; round up.

Answer: 14.

(c) Verify the boundary using $n-1$ and n. [2 marks]

The first probability is below 0.95 and the second is above it.

Answer: $n = 13$ gives 0.9450; $n = 14$ gives 0.9560.

10. Game A charges 2 dollars and pays 10 dollars with probability 0.2, otherwise 0 dollars. Game B charges 2 dollars and always returns 2 dollars. A player can choose either. [7 marks]

(a) Compare expected net gains. [2 marks]

A has expected prize $10(0.2) = 2$ and fee 2.

B returns exactly its fee.

Answer: Both are zero.

(b) Compare net-gain variances. [2 marks]

For A, gross $E(X^2) = 100(0.2) = 20$ and mean 2, giving variance 16.

Subtracting the fee leaves variance unchanged.

Answer: Game A: 16 dollars²; Game B: 0.

(c) For ten independent plays of A, find the probability of a strictly positive total net gain. [3 marks]

If K is the number of wins, total net gain is $10K - 20$.

Strictly positive means $K \geq 3$, with $K \sim \text{Bin}(10, 0.2)$.

$\sum_{k=3}^{10} \binom{10}{k} (0.2)^k (0.8)^{10-k} = 0.3222.$

Answer: 0.3222