Discrete distributions - Practice test 3

10 questions | 62 marks | 70 minutes

Apply the models to service times, insurance, tariffs and capacity decisions, with explicit assumptions.

Equivalent correct methods earn the corresponding marks.

1. A waiting time X is 1, 2 or 4 minutes with probabilities 0.3, 0.5 and 0.2. [6 marks]

(a) Find its expected waiting time. [2 marks]

$1(0.3)+2(0.5)+4(0.2)=2.1.$

Answer: 2.1 minutes.

(b) Find its variance and standard deviation. [2 marks]

$E(X^2)=0.3+2+3.2=5.5.$

Variance $=5.5-2.1^2=1.09.$

Answer: 1.09 min²; approximately 1.0440 minutes.

(c) What is $P(X>2)$? [2 marks]

Only the four-minute outcome exceeds two minutes.

Answer: 0.2.

2. A team records 100 daily counts: 30 days have count 0, 50 have count 1 and 20 have count 2. [6 marks]

(a) Use relative frequencies to estimate a distribution for one daily count. [2 marks]

Divide each count of days by 100.

Answer: Estimated probabilities 0.3, 0.5, 0.2 for 0, 1, 2.

(b) Find the mean under this estimated model. [2 marks]

$0(0.3)+1(0.5)+2(0.2)=0.9.$

Answer: 0.9.

(c) Does the model guarantee exactly 20 count-2 days in the next 100 days? Explain. [2 marks]

0.2 is an estimate of a probability, not a fixed schedule.

Future conditions may also differ from the recorded period.

Answer: No.

3. One insurance policy has claim X equal to 0 dollars, 500 dollars or 2000 dollars with probabilities 0.96, 0.03 and 0.01. The premium is 50 dollars. [6 marks]

(a) Find the expected claim. [2 marks]

$500(0.03)+2000(0.01)=15+20.$

Answer: 35 dollars.

(b) Find expected profit per policy before administrative costs. [2 marks]

Premium minus expected claim $=50-35.$

Answer: 15 dollars.

(c) Allowing a fixed administrative cost of 5 dollars per policy, find expected profit across 200 policies. [2 marks]

Expected profit per policy is $50-35-5=10.$

Multiply by 200; linearity of expectation is sufficient.

Answer: 2000 dollars.

4. A daily repair cost X, in tens of dollars, is 0, 2 or 6 with probabilities 0.5, 0.4 and 0.1. [6 marks]

(a) Find its mean and variance in the stated units. [2 marks]

$E(X) = 0.8 + 0.6 = 1.4.$

$E(X^2) = 1.6 + 3.6 = 5.2$; variance $= 5.2 - 1.96.$

Answer: Mean 1.4; variance 3.24.

(b) Find the standard deviation in dollars. [2 marks]

$\mathrm{SD}(X) = \sqrt{3.24} = 1.8$ tens of dollars.

Multiply by 10 to convert to dollars.

Answer: 18 dollars.

(c) A fixed call-out fee of 50 dollars is added. What happens to the dollar standard deviation? [2 marks]

A constant shift changes the mean, not the spread.

Answer: It remains 18 dollars.

5. A delivery tariff is $T = 4 + 2.5X$ dollars. The daily distance X has mean 3.2 km and variance 1.5 km². [6 marks]

(a) Find $E(T)$. [2 marks]

$4 + 2.5(3.2) = 12$.

Answer: 12 dollars.

(b) Find $\mathrm{Var}(T)$ and $\mathrm{SD}(T)$. [2 marks]

Variance $= 2.5^2(1.5) = 9.375$.

Take its square root.

Answer: Variance 9.375 dollars²; SD 3.0619 dollars.

(c) Explain why an increase of 1 dollar in the fixed tariff leaves $\mathrm{Var}(T)$ unchanged. [2 marks]

Distances retain their original spread and the multiplier is unchanged.

Answer: Every tariff outcome increases by the same amount.

6. Independent service times A and B have means 3 and 4 minutes and variances 0.25 and 1 min². [6 marks]

(a) Find the mean and variance of $A - B$. [2 marks]

$E(A - B) = 3 - 4$.

Independent variances add: $0.25 + 1$.

Answer: Mean -1 minute; variance 1.25 min².

(b) Does the negative mean prove that $A < B$ on every occasion? [2 marks]

A mean difference is not a statement about every paired outcome.

Answer: No.

(c) Can $P(A < B)$ be calculated uniquely from the supplied moments alone? Explain. [2 marks]

The means and variances do not specify the full distributions.

Answer: No.

7. A daily count X is 0, 1, 2 or 3 with probabilities 0.55, 0.25, 0.15 and 0.05. A flag B is 1 when $X \geq 2$ and 0 otherwise. [6 marks]

(a) Find $P(B = 1)$, $E(B)$ and $\mathrm{Var}(B)$. [2 marks]

$P(X \geq 2) = 0.15 + 0.05$.

The indicator is Bernoulli with $p = 0.2$.

Answer: 0.2, 0.2 and 0.16.

(b) For 40 independent comparable days, find the mean and variance of the number of flags. [2 marks]

Use $\mathrm{Bin}(40, 0.2)$.

$np = 8$ and $np(1 - p) = 6.4$.

Answer: Mean 8; variance 6.4.

(c) Name a plausible reason independence might fail. [2 marks]

Shared or persistent conditions can correlate the outcomes.

Answer: A fault persisting across successive days.

8. A day is equally likely to be quiet or busy. Conditional on the day type, two customer outcomes are independent. Each customer buys with probability 0.2 on a quiet day and 0.6 on a busy day. [7 marks]

(a) Find one customer's unconditional purchase probability. [2 marks]

$0.5(0.2) + 0.5(0.6) = 0.4$.

Answer: 0.4.

(b) Find the unconditional probability that both buy. [3 marks]

Average the conditional pair probabilities: $0.5(0.2^2) + 0.5(0.6^2) = 0.2$.

Answer: 0.2.

(c) Explain why $\mathrm{Bin}(2, 0.4)$ is not exact for the purchase count. [2 marks]

That binomial model predicts $0.4^2 = 0.16$ for both buying, not 0.2.

Answer: The shared day type creates unconditional dependence.

9. An independent-item fault model gives a fault probability of 0.08. Each batch contains 12 items. [6 marks]

(a) Find the probability of at most one faulty item. [2 marks]

Let X be $\text{Bin}(12, 0.08)$.

$\sum_{k=0}^{1} \binom{12}{k}(0.08)^k(0.92)^{12-k} = 0.7513.$

Answer: 0.7513

(b) Find the probability of at least two faulty items. [2 marks]

Subtract the unrounded result in (a) from 1.

Answer: 0.2487

(c) Among 300 batches, find the expected number containing at least two faults. [2 marks]

Multiply 300 by the probability in (b).

Answer: 74.6045

10. Twenty independent customers each require a service with probability 0.1. Capacity c is exceeded when more than c customers require it. [7 marks]

(a) Find the exceedance probabilities for capacities 3 and 4. [3 marks]

Use $X \sim \text{Bin}(20, 0.1)$.

Calculate $P(X \geq 4)$ and $P(X \geq 5)$, respectively.

Answer: $c = 3$: 0.1330; $c = 4$: 0.0432.

(b) Choose the smaller of these capacities that keeps exceedance probability below 0.05. [2 marks]

The capacity-3 probability is above 0.05; the capacity-4 probability is below it.

Answer: Capacity 4.

(c) Give one limitation of the decision. [2 marks]

A shared event could increase demand probabilities or make customers dependent.

Answer: It depends on stable independent customer demand with $p = 0.1$.