Discrete distributions - Practice test 2

10 questions | 64 marks | 70 minutes

Compare models, recover a distribution from cumulative information and examine the role of independence.

Equivalent correct methods earn the corresponding marks.

1. A bag contains two red and one blue counter. Two counters are selected fairly without replacement. X is the red count. [8 marks]

(a) Construct the distribution of X. [3 marks]

The three equally likely unordered pairs contain either both red counters or one red and the blue.

Two of the three pairs contain exactly one red.

Answer: $P(X=0)=0,\ P(X=1)=\frac{2}{3},\ P(X=2)=\frac{1}{3}$.

(b) Calculate $E(X)$. [2 marks]

$E(X)=1(\frac{2}{3})+2(\frac{1}{3})=\frac{4}{3}$.

Answer: $\frac{4}{3}$.

(c) Explain why a $\text{Bin}(2,\frac{2}{3})$ model has the correct mean but is not the correct distribution. [3 marks]

$\text{Bin}(2,\frac{2}{3})$ predicts $P(X=2)=\frac{4}{9}$ rather than $\frac{1}{3}$.

Answer: The mean matches, but replacement is absent and trials are dependent.

2. X takes only -2, 0 and 5. Its cumulative probabilities at these values are 0.2, 0.7 and 1. [6 marks]

(a) Find $P(X=0)$. [2 marks]

Subtract the previous cumulative value: $0.7-0.2$.

Answer: 0.5.

(b) Find $P(-1<X<5)$. [2 marks]

Only $X=0$ lies strictly between -1 and 5.

Answer: 0.5.

(c) Find $F(3)$. [2 marks]

No probability mass occurs between 0 and 5, so the cumulative value stays 0.7.

Answer: 0.7.

3. X takes values 0, 2 and 8 with probabilities a, 0.5 and $0.5-a$. Its expected value is 3. [6 marks]

(a) Find a. [3 marks]

$E(X)=2(0.5)+8(0.5-a)=5-8a$.

Solve $5-8a=3$.

Answer: $a=0.25$.

(b) Find $\text{Var}(X)$. [2 marks]

$E(X^2)=4(0.5)+64(0.25)=18$.

$\text{Var}(X)=18-3^2=9$.

Answer: 9.

(c) Check that all probabilities are valid. [1 mark]

Each is non-negative and their sum is 1.

Answer: They are 0.25, 0.5, 0.25.

4. Option A pays 0 or 2 with equal probabilities. Option B always pays 1. [6 marks]

(a) Show that their means are equal. [2 marks]

For A, $E(A) = 0(0.5) + 2(0.5) = 1$; B is constant at 1.

Answer: Both means are 1.

(b) Compare their variances. [2 marks]

$E(A^2) = 2$, giving variance $2 - 1 = 1$.

A constant has zero variance.

Answer: $\text{Var}(A) = 1;\ \text{Var}(B) = 0$.

(c) Which option avoids the possibility of receiving zero? Explain. [2 marks]

A pays zero with probability 0.5; B always pays 1.

Answer: Option B.

5. X takes 0, 1 and 3 with probabilities 0.1, 0.2 and 0.7. Let $Y = 10 - 2X$. [7 marks]

(a) Give the values and probabilities of Y. [2 marks]

Apply $10 - 2x$ to each value and retain its probability.

Answer: $Y = 10, 8, 4$ with probabilities 0.1, 0.2, 0.7.

(b) Find $P(Y \geq 8)$. [2 marks]

Use $Y = 10$ or $Y = 8$, hence $0.1 + 0.2$.

Answer: 0.3.

(c) Find the mean and variance of Y. [3 marks]

$E(X) = 2.3;\ E(X^2) = 6.5;\ \text{Var}(X) = 6.5 - 2.3^2 = 1.21$.

$E(Y) = 10 - 2(2.3) = 5.4;\ \text{Var}(Y) = 4(1.21)$.

Answer: Mean 5.4; variance 4.84.

6. Independent success indicators B and C have success probabilities 0.2 and 0.6. Set $S = B + C$. [7 marks]

(a) Find the distribution of S. [3 marks]

No success: $0.8(0.4)$. Both: $0.2(0.6)$.

One success: $0.2(0.4) + 0.8(0.6)$.

Answer: $P(S = 0) = 0.32$; $P(S = 1) = 0.56$; $P(S = 2) = 0.12$.

(b) Find its mean and variance. [2 marks]

Add Bernoulli means: $0.2 + 0.6$.

Add independent variances: $0.16 + 0.24$.

Answer: Mean 0.8; variance 0.4.

(c) Why is S not $\text{Bin}(2, 0.4)$? [2 marks]

$\text{Bin}(2, 0.4)$ would have variance $2(0.4)(0.6) = 0.48$, not 0.4.

Answer: The success probabilities are different.

7. A fair coin is tossed once. $B = 1$ for heads and C is defined to equal B. Let $S = B + C$. [6 marks]

(a) Construct the distribution of S and find $\text{Var}(S)$. [2 marks]

$E(S) = 1$ and $E(S^2) = 2$, so $\text{Var}(S) = 1$.

Answer: S is 0 or 2 with equal probabilities; variance 1.

(b) Why is adding the two Bernoulli variances invalid here? [2 marks]

They are always equal; their sum variance is not $0.25 + 0.25 = 0.5$.

Answer: B and C are dependent.

(c) Does $E(S) = E(B) + E(C)$ still hold? Explain. [2 marks]

Additivity of expectation does not require independence.

Answer: Yes; both sides equal 1.

8. A group has 3 marked and 17 unmarked items. Two are sampled without replacement. [6 marks]

(a) Find the exact probability that both are marked. [2 marks]

$(\frac{3}{20})(\frac{2}{19}) = \frac{3}{190}$.

Answer: $\frac{3}{190} \approx 0.0158$.

(b) Find the probability predicted by $\text{Bin}(2, 0.15)$ for two marked items. [2 marks]

The independent model gives 0.15^2.

Answer: 0.0225.

(c) Explain the source of the difference. [2 marks]

Given a marked first item, the next marked probability is $\frac{2}{19}$, not $\frac{3}{20}$.

Answer: The second draw changes the available composition.

9. $X \sim \text{Bin}(6, 0.4)$. Give probabilities to four decimal places. [6 marks]

(a) Find $P(1 \leq X \leq 3)$. [2 marks]

$\sum_{k=1}^{3} \binom{6}{k}(0.4)^k(0.6)^{6-k} = 0.7741.$

Answer: 0.7741

(b) Find $P(X > 3)$. [2 marks]

Use $1 - P(X \leq 3)$.

$\sum_{k=4}^{6} \binom{6}{k}(0.4)^k(0.6)^{6-k} = 0.1792.$

Answer: 0.1792

(c) Among 80 comparable groups, how many are expected to have more than three successes? [2 marks]

Multiply the unrounded probability in (b) by 80.

Answer: 14.3360

10. A device works if at least 3 of its 4 independent components work. Each component works with probability p. [6 marks]

(a) Find the device reliability for $p = 0.9$. [2 marks]

Let X be $\text{Bin}(4, 0.9)$.

$P(X \geq 3) = 4(0.9^3)(0.1) + 0.9^4.$

Answer: 0.9477.

(b) Find the reliability for $p = 0.95$. [2 marks]

Use $\text{Bin}(4, 0.95)$: $4(0.95^3)(0.05) + 0.95^4$.

Answer: 0.98598125.

(c) Which component choice meets a device target of at least 0.98? [2 marks]

$0.9477 < 0.98$ whereas $0.98598125 \geq 0.98$.

Answer: The $p = 0.95$ components.