Discrete distributions - Practice test 1

10 questions | 58 marks | 70 minutes

Build confidence with tables, cumulative probabilities, summary measures and an ordinary binomial model.

Equivalent correct methods earn the corresponding marks.

1. X has the probability distribution shown. [6 marks]

Distribution of X

x	0	1	2	4
$P(X=x)$	0.2	0.3	k	0.1

(a) Find k. [2 marks]

Probabilities sum to 1: $k = 1 - 0.2 - 0.3 - 0.1$.

Answer: $k = 0.4$.

(b) Calculate $E(X)$. [2 marks]

$E(X) = 0(0.2) + 1(0.3) + 2(0.4) + 4(0.1) = 1.5$.

Answer: 1.5.

(c) Find $P(X \geq 2)$. [2 marks]

Add probabilities for 2 and 4: $0.4 + 0.1$.

Answer: 0.5.

2. X is integer-valued and has $F(0)=0.1$, $F(1)=0.4$, $F(2)=0.75$ and $F(3)=1$. It takes no other values. [6 marks]

(a) Find $P(X=2)$. [2 marks]

The jump at 2 is $F(2)-F(1)=0.75-0.4$.

Answer: 0.35.

(b) Find $P(1\le X\le 2)$. [2 marks]

Subtract $F(0)$ from $F(2)$: $0.75-0.1$.

Answer: 0.65.

(c) Find $P(X>1)$. [2 marks]

Use $1-F(1)=1-0.4$.

Answer: 0.6.

3. A game pays 0 dollars, 5 dollars or 20 dollars with probabilities 0.6, 0.3 and 0.1. The entry fee is 4 dollars. [6 marks]

(a) Find the expected gross prize. [2 marks]

$0(0.6)+5(0.3)+20(0.1)=3.5$.

Answer: 3.50 dollars.

(b) Find the fair fee and expected net gain at the actual fee. [2 marks]

A fair fee equals the gross expected prize.

Subtract the actual fee: $3.50-4.00=-0.50$.

Answer: Fair fee 3.50 dollars; expected net gain -0.50 dollars.

(c) Find expected total net gain over 200 plays. [2 marks]

Add expected gains: $200(-0.50)=-100$.

Answer: -100 dollars.

4. X takes values -1, 1 and 3 with probabilities $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$. [5 marks]

(a) Find $E(X)$ and $E(X^2)$. [2 marks]

$E(X) = -\frac{1}{4} + \frac{1}{2} + \frac{3}{4} = 1.$

$E(X^2) = \frac{1}{4} + \frac{1}{2} + \frac{9}{4} = 3.$

Answer: 1 and 3.

(b) Find $\mathrm{Var}(X)$ and $\mathrm{SD}(X)$. [2 marks]

$\mathrm{Var}(X) = 3 - 1^2 = 2.$

SD is the positive square root of variance.

Answer: $2;\ \sqrt{2} \approx 1.4142.$

(c) Explain why the variance is not expressed in the same units as X. [1 mark]

If X has units u, the variance has units u^2.

Answer: The deviations are squared.

5. A score X has mean 8 and variance 2. A conversion uses $Y = 3X - 5$. [6 marks]

(a) Find $E(Y)$. [2 marks]

$3(8) - 5 = 19$.

Answer: 19.

(b) Find $\mathrm{Var}(Y)$ and $\mathrm{SD}(Y)$. [2 marks]

Variance multiplies by 3^2: $9(2) = 18$.

The fixed shift does not affect spread.

Answer: 18; $\sqrt{18} \approx 4.2426$.

(c) If the subtraction of 5 is removed, what happens to the mean and variance? [2 marks]

A fixed shift changes every outcome by the same amount.

Answer: The mean increases by 5 to 24; variance remains 18.

6. Independent X and Y have means 4 and 3 and variances 5 and 2. [5 marks]

(a) Find the mean and variance of $2X + Y$. [2 marks]

$E(2X + Y) = 2(4) + 3$.

$\mathrm{Var}(2X + Y) = 4(5) + 2$ by independence.

Answer: Mean 11; variance 22.

(b) Find the mean and variance of $X - Y$. [2 marks]

$E(X - Y) = 4 - 3$.

$\mathrm{Var}(X - Y) = 5 + 2$.

Answer: Mean 1; variance 7.

(c) Which calculation in part (b) still works if independence is not known? [1 mark]

Linearity of expectation does not need independence; adding variances does here.

Answer: The mean calculation.

7. A check passes with probability 0.8. Let $B = 1$ for a failure and $B = 0$ for a pass. [6 marks]

(a) State the distribution and mean of B. [2 marks]

Success for this indicator is failure of the check.

Answer: $P(B = 1) = 0.2$, $P(B = 0) = 0.8$; mean 0.2.

(b) Find $\mathrm{Var}(B)$. [2 marks]

$0.2(0.8) = 0.16$.

Answer: 0.16.

(c) Find the mean and variance of the score $4B - 1$. [2 marks]

Mean $= 4(0.2) - 1$.

Variance $= 4^2(0.16)$.

Answer: Mean -0.2; variance 2.56.

8. A biased coin has $P(\text{heads}) = 0.3$ and is tossed independently five times. X counts tails. [5 marks]

(a) State a binomial model for X and justify its parameters. [3 marks]

There are five fixed independent trials, tails/no tails categories and common tails probability 0.7.

Answer: $X \sim \mathrm{Bin}(5, 0.7)$.

(b) A different process changes the coin bias after each toss. Explain which binomial condition may fail. [2 marks]

Changing the bias can make the tail probability different on successive trials.

Answer: The common success probability condition.

9. $X \sim \text{Bin}(8, 0.3)$. Give probabilities to four decimal places. [6 marks]

(a) Find $P(X = 2)$. [2 marks]

$\binom{8}{2}(0.3)^2(0.7)^6 = 0.29647548.$

Answer: 0.2965

(b) Find $P(X \geq 2)$. [2 marks]

Use $1 - P(X = 0) - P(X = 1)$.

$\sum_{k=2}^{8} \binom{8}{k}(0.3)^k(0.7)^{8-k} = 0.7447.$

Answer: 0.7447

(c) State $E(X)$ and explain why it need not be a possible outcome. [2 marks]

$E(X) = 8(0.3) = 2.4$; a single count is still an integer.

Answer: 2.4; it is a long-run mean.

10. A binomial variable has exact mean 12 and exact variance 9. [7 marks]

(a) Find n and p. [3 marks]

$1 - p = \frac{9}{12} = 0.75$, hence $p = 0.25$.

$n = \frac{12}{0.25} = 48.$

Answer: $n = 48,\ p = 0.25.$

(b) Find the probability of zero successes, using scientific notation if useful. [2 marks]

$P(X = 0) = 0.75^{48}.$

Answer: 1.00679×10^{-06}

(c) Does the mean force each set of 48 trials to contain 12 successes? Explain. [2 marks]

The mean averages counts over many repeated sets; any count from 0 to 48 remains possible for $0 < p < 1$.

Answer: No.