Integration - Applications and reasoning test

10 questions | 49 marks | 75 minutes

A harder original applications paper testing choice of method, model assumptions and mathematical explanation.

Equivalent correct methods earn the corresponding marks.

1. A parameter appears in an integral. [4 marks]

(a) If $\int_0^2 (kx+1)\,dx = 8$, find k. [2 marks]

$2k+2=8$ gives 3.

Answer: 3

(b) Find the average value of $kx+1$ over $[0,2]$ for this k. [2 marks]

Divide the given integral 8 by interval length 2.

Answer: 4

2. A student proposes $F(x)=\ln(2x+1)$ as an antiderivative of $\frac{1}{2x+1}$ on $x>0$. [4 marks]

(a) Identify and explain the error. [2 marks]

The chain-rule factor has not been undone.

Answer: $F'=\frac{2}{2x+1}$, twice the required integrand.

(b) Correct the antiderivative and evaluate the integral from 0 to 1. [2 marks]

Divide by 2 then evaluate.

Answer: $\frac{1}{2}\ln(2x+1)+C$; definite value $\frac{1}{2}\ln 3$.

3. A quantity satisfies $N'(t) = 4e^{2t}$ and $N(0) = 7$. [5 marks]

(a) Find $N(t)$. [2 marks]

$N = 2e^{2t} + C;\ C = 5.$

Answer: $2e^{2t} + 5$

(b) Find when $N = 13$. [3 marks]

$e^{2t} = 4$ gives $t = \frac{\ln 4}{2} = \ln 2.$

Answer: $t = \ln 2$

4. A rate is recorded at 0,1 and 2 hours. [6 marks]

Time (h)	0	1	2
Rate (units/h)	2	5	10

(a) Estimate the accumulated change by trapeziums. [2 marks]

$\frac{2+5}{2} + \frac{5+10}{2} = 11.$

Answer: 11 units

(b) Can the true change be determined exactly from these values alone? Explain. [2 marks]

Different continuous curves can pass through the same samples and have different integrals.

Answer: No.

(c) Name useful additional information. [2 marks]

More samples can improve an estimate but do not alone prove exactness.

Answer: An explicit rate formula or justified between-sample model.

5. $A(x)=\int_0^{x^2}(t+1)\,dt$. [4 marks]

(a) Find $A'(x)$. [2 marks]

Endpoint substitution times endpoint derivative.

Answer: $2x(x^2+1)$

(b) Find all stationary inputs and explain why there are no others. [2 marks]

$x^2+1>0$ for real x, so $2x=0$.

Answer: $x=0$

6. The graphs $y=x$ and $y=-x$ are considered on $[-2,1]$. [5 marks]

(a) Find where the upper boundary changes. [2 marks]

The two lines intersect there.

Answer: At $x=0$.

(b) Find the geometric area. [3 marks]

$\int_{-2}^{0}(-2x)\,dx+\int_0^1 2x\,dx=4+1=5$.

Answer: 5

7. The graph $y=x^2-1$ lies below the axis on $[-1,1]$. [6 marks]

(a) Find its geometric area on this interval. [2 marks]

$\int_{-1}^{1}(1-x^2)\,dx=\frac{4}{3}$.

Answer: $\frac{4}{3}$

(b) Find $b>1$ such that its signed integral from -1 to b is 0. [4 marks]

$[\frac{x^3}{3}-x]_{-1}^{b}=0$ gives $b^3-3b-2=0$.

$(b-2)(b+1)^2=0$; use $b>1$.

Answer: $b=2$

8. A tank starts with 15 L and has net rate $6-2t$ L/min for $0 \le t \le 5$. [5 marks]

(a) Find its maximum volume. [2 marks]

The rate changes $+$ to $-$ at 3; change to 3 is 9 L.

Answer: 24 L at 3 min.

(b) Find its final volume. Explain why the total inflow cannot be determined from this net-rate model. [3 marks]

Net change to 5 is 5 L; the net-rate model alone does not separate simultaneous inflow/outflow.

Answer: 20 L.

9. The temperature model is $T(t) = t^2 + 2$°C over $[0, 3]$. [4 marks]

(a) Find the time-average temperature. [2 marks]

$\frac{1}{3}\int_0^3 (t^2+2)\,dt = \frac{9+6}{3} = 5.$

Answer: 5°C

(b) Compare with the average of endpoint temperatures. [2 marks]

Endpoints 2 and 11 give 6.5; curvature means this is not the time average.

Answer: Endpoint average 6.5°C.

10. A particle has $v(t) = t^2 - 4t + 3$ m/s on $[0, 4]$, initial position 2 m. [6 marks]

(a) Find its final position. [2 marks]

Displacement $\frac{4}{3}$ m; add 2 m.

Answer: $\frac{10}{3}$ m

(b) Find the total distance and explain why it is larger than the magnitude of displacement. [4 marks]

The particle reverses at 1 and 3; distances $\frac{4}{3}+\frac{4}{3}+\frac{4}{3}$ add while signed changes partly cancel.

Answer: 4 m