Logarithmic functions - Practice test 6

10 questions | 61 marks | 70 minutes

A more demanding paper combining precise domains, parameter choices and mathematical reasoning.

Equivalent correct methods earn the corresponding marks.

1. Solve and justify the base restrictions. [5 marks]

(a) Find b if $\log_b\left(\frac{1}{16}\right)=4$. [2 marks]

$b^4=\frac{1}{16}$ with $b>0$; the positive fourth root is $\frac{1}{2}$.

Answer: $b=\frac{1}{2}$.

(b) Explain why base 1 cannot define a logarithm on all positive inputs. [3 marks]

An inverse function requires a unique output for every allowed input.

Answer: $1^y=1$ for every y, so inputs other than 1 are never produced and input 1 has no unique exponent.

2. Consider $F(x)=\ln(x-1)+\ln(x+3)$. [6 marks]

(a) State its domain and solve $F(x)=\ln(12)$. [4 marks]

$(x-1)(x+3)=12$ gives $x^2+2x-15=0$; its roots are $3,\ -5$. Only 3 is valid.

Answer: Domain $x>1$; solution $x=3$.

(b) Explain why solving the combined equation may introduce an inadmissible branch. [2 marks]

Original-domain checking removes the negative branch.

Answer: The combined positive-product condition also allows $x<-3$, whereas both original inputs must be positive.

3. Solve $e^x + 3e^{-x} = 4$. [5 marks]

(a) Find all possible positive values of $u = e^x$. [3 marks]

$u + \frac{3}{u} = 4$ gives $u^2 - 4u + 3 = 0$.

Answer: $u = 1$ or 3.

(b) Find all x and decide which is larger. [2 marks]

Apply the natural logarithm to each positive u.

Answer: $x = 0$ and $x = \ln 3$; $\ln 3 > 0$.

4. A family is $f_a(x) = \ln(x - a) + 2a$. [6 marks]

(a) Find the domain and x-intercept in terms of a. [3 marks]

Solve $\ln(x - a) = -2a$.

Answer: $x > a$; intercept $(a + e^{-2a}, 0)$.

(b) Find the inverse function and state its range. [3 marks]

Solve $y - 2a = \ln(x - a)$ for x and interchange names.

Answer: $f_a^{-1}(x) = a + e^{x - 2a}$; range (a, ∞).

5. Let $g(x) = \ln x - kx$ for $x > 0$ and $k > 0$. [7 marks]

(a) Find the stationary point in terms of k and classify it. [4 marks]

$g' = \frac{1}{x} - k$ and $g'' = -\frac{1}{x^2} < 0$.

Answer: $x = \frac{1}{k}$, $y = -\ln k - 1$; a maximum.

(b) Find k so that the maximum value is zero. [3 marks]

$-\ln k - 1 = 0$ implies $\ln k = -1$.

Answer: $k = e^{-1}$.

6. Consider $h(x)=\ln(\frac{x+1}{x-1})$. [6 marks]

(a) Find its real domain. [3 marks]

The ratio must be positive, and its denominator must be nonzero; a sign chart gives the two intervals.

Answer: $x<-1$ or $x>1$.

(b) Differentiate h on its entire domain without splitting it into invalid logarithms. [3 marks]

For $u=\frac{x+1}{x-1}$, $u'=-\frac{2}{(x-1)^2}$. Then $\frac{u'}{u}=-\frac{2}{(x-1)(x+1)}$.

Answer: $h'(x)=-\frac{2}{x^2-1}$.

7. The curve is $y=\ln(2+x^2)$. [7 marks]

(a) Find the tangent at $x=1$. [3 marks]

The derivative is $\frac{2x}{2+x^2}$, so the gradient is $\frac{2}{3}$.

Answer: $y-\ln 3=\frac{2}{3}(x-1)$.

(b) Find the two points at which the tangent gradient equals $\frac{1}{2}$. [4 marks]

Solve $\frac{2x}{2+x^2}=\frac{1}{2}$, giving $x^2-4x+2=0$. Use $2+x^2=4x$ at these roots.

Answer: $x=2\pm\sqrt{2}$; the corresponding y-values are $\ln(8\pm4\sqrt{2})$.

8. A response is $R(t)=A+B\ln(1+\frac{t}{c})$ with $B,c>0$ and $t\geq0$. Time is in hours. [7 marks]

(a) Show that its derivative is $\frac{B}{c+t}$ and give the rate units. [3 marks]

Apply the chain rule: $\frac{B(\frac{1}{c})}{1+\frac{t}{c}}$.

Answer: $R'(t)=\frac{B}{c+t}$, response units per hour.

(b) If the initial rate is 6 and the rate at $t=4$ is 2, find B and c. [4 marks]

$\frac{B}{c}=6$ gives $B=6c$. Then $\frac{B}{c+4}=2$ gives $6c=2c+8$.

Answer: $c=2$, $B=12$.

9. A model score is $S(x) = 10\log_{10}\left(\frac{x}{x_0}\right)$ for positive x and x_0. [6 marks]

(a) A change of 7 scale units occurs. Find the exact multiplier of x. [3 marks]

$7 = 10\log_{10}\left(\frac{x_2}{x_1}\right)$, so divide by 10 and exponentiate.

Answer: $10^{0.7}$.

(b) Show that adding equal amounts to x does not generally give equal changes in S. [3 marks]

Subtract the scales at $x + d$ and x, for positive inputs.

Answer: The change for adding d is $10\log_{10}\left(1 + \frac{d}{x}\right)$, which depends on x.

10. A proposed simplification and an asymptote need careful interpretation. [6 marks]

(a) Compare the functions $p(x) = \ln(x^2 - 1)$ and $q(x) = \ln(x - 1) + \ln(x + 1)$. Where do they agree and where is only p defined? [3 marks]

p needs a positive product, while q needs both factors positive.

Answer: They agree for $x > 1$; only p is defined for $x < -1$.

(b) For p, find its derivative and the limiting behaviour as $x \to -1^-$ and $x \to 1^+$. [3 marks]

The positive input tends to zero at each boundary, so its natural logarithm tends to negative infinity.

Answer: $p' = \frac{2x}{x^2 - 1}$; both limits are $-\infty$.