Logarithmic functions - Practice test 3

10 questions | 56 marks | 70 minutes

A mixed paper connecting inverse functions, differentiation and supplied mathematical models.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = 2e^{3x}$. [5 marks]

(a) Find its inverse function. [3 marks]

$\frac{y}{2} = e^{3x}$ gives $x = \frac{\ln(\frac{y}{2})}{3}$; swap names.

Answer: $f^{-1}(x) = \frac{1}{3}\ln(\frac{x}{2})$, $x > 0$.

(b) Explain the range of the inverse and evaluate it at $2e$. [2 marks]

The original domain is all real numbers; $\ln e = 1$.

Answer: Range $\mathbb{R}$; $f^{-1}(2e) = \frac{1}{3}$.

2. Apply logarithm laws. [5 marks]

(a) For $x > 1$, expand $\ln(\frac{(x-1)^3}{\sqrt{x+2}})$. [3 marks]

The assumed domain makes both logarithm inputs positive.

Answer: $3\ln(x-1) - \frac{1}{2}\ln(x+2)$.

(b) Simplify $\log_4 2 + \log_4 8$. [2 marks]

Combine the product: $\log_4 16 = 2$.

Answer: 2

3. Solve $[\ln x]^2 - 3\ln x + 2 = 0$. [5 marks]

(a) State a substitution and find the possible logarithm values. [3 marks]

Factor $u^2 - 3u + 2 = (u-1)(u-2)$.

Answer: $u = \ln x$; $u = 1$ or 2.

(b) Find all x and explain why neither is rejected. [2 marks]

Both exponentials are positive and satisfy the original domain $x > 0$.

Answer: $x = e$ or e^2.

4. Consider two exponential expressions. [5 marks]

(a) Solve $e^{2x} + 2e^x - 8 = 0$. [3 marks]

$u = e^x > 0$ gives $(u+4)(u-2) = 0$. Reject $u = -4$.

Answer: $x = \ln 2$

(b) Explain why $e^{2x} + 2e^x + 8 = 0$ has no real solution without using the discriminant. [2 marks]

$e^x > 0$ and $e^{2x} > 0$.

Answer: Every term is positive for real x, so their sum cannot be zero.

5. A deposit is modelled by $A(n) = 1000(1.04)^n$ after n complete years. [6 marks]

(a) Extend the model to real values of n and find the time at which A reaches 1500. [3 marks]

$1.04^n = 1.5$; take logarithms.

Answer: $n = \frac{\ln 1.5}{\ln 1.04} \approx 10.3$ years.

(b) Find the first complete year with A at least 1500, justifying the rounding direction. [3 marks]

The threshold time is approximately 10.338 years, so round up; at year 10 it is below 1500.

Answer: Year 11.

6. A transformed graph has form $g(x) = a\ln(x-1) + b$. [6 marks]

(a) Use $g(2) = 4$ and $g(1+e) = 7$ to find a and b. [3 marks]

$\ln 1 = 0$ gives $b = 4$; $\ln e = 1$ gives $a + 4 = 7$.

Answer: $a = 3$, $b = 4$.

(b) Find its x-intercept exactly and its vertical asymptote. [3 marks]

$3\ln(x-1) + 4 = 0$ gives the intercept; the input vanishes at the asymptote.

Answer: $(1 + e^{-\frac{4}{3}}, 0)$; $x = 1$.

7. Let $f(x) = \frac{\ln x}{x}$ for $x > 0$. [6 marks]

(a) Find its derivative. [3 marks]

Use the quotient rule: $\frac{(\frac{1}{x})x - \ln x}{x^2}$.

Answer: $f'(x) = \frac{1 - \ln x}{x^2}$.

(b) Find and classify its stationary point. [3 marks]

The numerator is positive before e and negative after e; the denominator is positive.

Answer: $(e, \frac{1}{e})$, a maximum.

8. Let $f(x) = \ln(1 + x^2)$. [7 marks]

(a) Find the first and second derivatives. [4 marks]

Use the chain rule followed by the quotient rule.

Answer: $f' = \frac{2x}{1+x^2}$; $f'' = \frac{2(1-x^2)}{(1+x^2)^2}$.

(b) Find the inflection points and verify a change of concavity. [3 marks]

The second derivative changes sign at both $x = -1$ and $x = 1$.

Answer: $(-1, \ln 2)$ and $(1, \ln 2)$.

9. A supplied response scale is $S = 5\ln(\frac{Q}{Q_0})$ for $Q, Q_0 > 0$. [5 marks]

(a) Find $\frac{Q}{Q_0}$ when $S = 15$. [2 marks]

$\ln(\frac{Q}{Q_0}) = 3$; exponentiate.

Answer: $\frac{Q}{Q_0} = e^3$.

(b) If Q doubles, find the exact change in S and explain why it does not depend on Q_0. [3 marks]

Subtract scales: $5[\ln(\frac{2Q}{Q_0}) - \ln(\frac{Q}{Q_0})] = 5\ln 2$.

Answer: $5\ln 2$.

10. Two proposed growth models are $F(t) = 10 + 4\ln(t+1)$ and $G(t) = 10 + 2t$ for $t \geq 0$. [6 marks]

(a) Show they have equal initial values but different initial rates. [3 marks]

Differentiate F to get $\frac{4}{t+1}$.

Answer: $F(0) = G(0) = 10$; $F'(0) = 4$, $G'(0) = 2$.

(b) At what time are their instantaneous rates equal? Does equal rate imply equal value there? [3 marks]

Solve $\frac{4}{t+1} = 2$ and substitute into both models.

Answer: $t = 1$; no, $F(1) = 10 + 4\ln 2 \neq 12 = G(1)$.