Differentiation practice test 6: synthesis and applications

10 questions | 76 marks | 75 minutes

Original chapter assessment with linked reasoning and applications. Show working, state relevant domains and retain exact values until a numerical answer is requested.

Equivalent correct methods earn the corresponding marks.

1. Differentiate efficiently, stating any excluded inputs. [5 marks]

(a) $f(x)=\frac{x^2-1}{x+1}$. [2 marks]

Simplify to $x-1$ only where the original denominator is nonzero.

Answer: $f'=1$ for $x\neq -1$

(b) $g(x)=(x^2+1)^3e^{-x}$. [3 marks]

Use product and chain rules.

Factor the common exponential and power.

Answer: $g'=e^{-x}(x^2+1)^2(6x-x^2-1)$

2. A temperature model is $T(t)=18+6\sin[\frac{\pi(t-4)}{12}]$ degrees Celsius, for $0\le t\le 24$ hours. [9 marks]

(a) Find the maximum and minimum temperatures and their times. [3 marks]

The sine is 1 at angle $\frac{\pi}{2}$ and -1 at $\frac{3\pi}{2}$.

Solve for t within the stated interval.

Answer: Maximum 24 degrees Celsius at $t=10$; minimum 12 degrees Celsius at $t=22$.

(b) Find when warming is fastest, and the rate. [3 marks]

$T'=(\frac{\pi}{2})\cos[\frac{\pi(t-4)}{12}]$.

Its maximum occurs when the cosine is 1.

Answer: $t=4$ h; $\frac{\pi}{2}$ degrees Celsius per hour

(c) Find both times when the temperature is 20 degrees Celsius. [3 marks]

Solve sine equal to $\frac{1}{3}$.

Use the two permitted angles in the displayed period.

Answer: $4+(\frac{12}{\pi})\arcsin(\frac{1}{3})$ and $16-(\frac{12}{\pi})\arcsin(\frac{1}{3})$ hours

3. Let $f(x) = x^3 - 3ax$, with $a > 0$. [7 marks]

(a) Find stationary coordinates in terms of a. [3 marks]

$f' = 3(x^2 - a)$; substitute $x = \pm\sqrt{a}$ into f.

Answer: $(-\sqrt{a}, 2a^{\frac{3}{2}})$ and $(\sqrt{a}, -2a^{\frac{3}{2}})$

(b) Classify them. [2 marks]

$f'' = 6x$ is respectively negative and positive.

Answer: The negative-input point is a maximum; the positive-input point is a minimum.

(c) Their vertical separation is 32. Find a. [2 marks]

$4a^{\frac{3}{2}} = 32$ gives $a^{\frac{3}{2}} = 8$.

Answer: $a = 4$

4. A cooling object follows $T(t) = 20 + 80e^{-kt}$ degrees Celsius, with t in minutes. After 5 minutes its temperature is 60 degrees Celsius. [8 marks]

(a) Find k exactly. [3 marks]

$40 = 80e^{-5k}$, so $e^{-5k} = \frac{1}{2}$.

Answer: $k = \frac{\ln 2}{5}$ per minute

(b) Find the initial rate of temperature change. [3 marks]

$T' = -80ke^{-kt}$; substitute zero and the value of k.

Answer: $-16\ln 2 \approx -11.1$ degrees Celsius per minute

(c) Find when its temperature reaches 30 degrees Celsius. [2 marks]

$10 = 80e^{-kt}$ gives $kt = \ln 8 = 3\ln 2$.

Answer: 15 minutes.

5. The curve $y=\frac{x^2+ax+b}{x-1}$ has oblique asymptote $y=x+3$ and y-intercept -4. [9 marks]

(a) Find a and b. [3 marks]

Polynomial division gives oblique asymptote $y=x+a+1$.

At zero the function value is $-b$.

Answer: $a=2,\ b=4$.

(b) Find stationary coordinates. [4 marks]

The function is $x+3+\frac{7}{x-1}$.

The derivative $1-\frac{7}{(x-1)^2}$ is zero when $(x-1)^2=7$.

Answer: $(1-\sqrt{7},4-2\sqrt{7})$ and $(1+\sqrt{7},4+2\sqrt{7})$

(c) Classify them. [2 marks]

The second derivative is $\frac{14}{(x-1)^3}$, negative left of 1 and positive right.

Answer: The left point is a local maximum; the right is a local minimum.

6. A closed cylindrical tank has volume 200π cubic metres. Material for its top and base costs twice as much per square metre as material for its curved side. [7 marks]

(a) Give a cost function proportional to the actual cost, in terms of radius r. [3 marks]

Use $h=\frac{200}{r^2}$.

Weight the combined top/base area by 2 and add side area.

Answer: $C(r)=4\pi r^2+\frac{400\pi}{r},\ r>0$

(b) Find the optimum relation between height and radius. [4 marks]

The derivative equation is $8\pi r-\frac{400\pi}{r^2}=0$.

Thus $r^3=50$ and $h=\frac{200}{r^2}=4r$.

The cost tends to infinity at both extremes, so the stationary minimum is absolute.

Answer: $r=\sqrt[3]{50}$ m and $h=4r$

7. A particle has $s(t) = \frac{t^4}{4} - 2t^3 + \frac{9t^2}{2} - 4t$ metres, $0 \le t \le 5$ seconds. [9 marks]

(a) Factorise the velocity and find its rest times. [3 marks]

Differentiate position and factor the resulting cubic.

Answer: $v = (t-1)^2(t-4)$; rest at 1 s and 4 s.

(b) Which rest time is a reversal? [2 marks]

The squared factor at 1 does not change sign.

The velocity changes negative-to-positive at 4.

Answer: Only $t = 4$ s.

(c) Find displacement and total distance. [4 marks]

Positions at $0, 4, 5$ are $0,\ -8,\ -\frac{5}{4}$.

The unchanged direction at 1 needs no extra distance segment.

Distance is $8 + \frac{27}{4} = \frac{59}{4}$.

Answer: Displacement $-\frac{5}{4}$ m; total distance $\frac{59}{4}$ m

8. A rectangle in the first quadrant has its upper-right corner on $y = 12e^{-x}$ and its lower-left corner at the origin. Its sides are parallel to the coordinate axes. [6 marks]

(a) Form the area function and find its derivative. [3 marks]

Area is width times height.

Use the product rule.

Answer: $A = 12xe^{-x}$; $A' = 12e^{-x}(1-x)$

(b) Find and justify its maximum area. [3 marks]

The derivative changes positive-to-negative at 1.

The area tends to zero as x tends to zero or infinity.

Answer: $\frac{12}{e}$ square units at width 1

9. A smooth function satisfies $f'(x) = (x^2 - 1)e^{-x}$. [8 marks]

(a) Find and classify the stationary inputs. [3 marks]

The exponential is positive and $x^2 - 1$ changes signs positive, negative, positive.

Answer: Maximum at $x = -1$; minimum at $x = 1$.

(b) Find the inputs where concavity changes. [3 marks]

$f'' = e^{-x}(-x^2 + 2x + 1)$.

The quadratic changes sign at both roots.

Answer: $1 - \sqrt{2}$ and $1 + \sqrt{2}$

(c) Can the stationary heights be determined from the supplied information alone? [2 marks]

An arbitrary added constant leaves the derivative unchanged.

Answer: No.

10. A product has demand price $p(q) = 80 - 0.5q$ dollars per unit and cost $C(q) = 20q + 200$ dollars, where $0 \le q \le 160$. [8 marks]

(a) Form profit as a function of output. [2 marks]

Revenue is $qp(q)$; subtract the cost.

Answer: $\Pi(q) = 60q - 0.5q^2 - 200$

(b) Find the profit-maximising output, price and profit. [4 marks]

The derivative is $60 - q$, zero at 60.

The quadratic is concave down.

Substitute into price and profit.

Answer: 60 units; price 50 dollars per unit; profit 1600 dollars.

(c) A student proposes 80 units by maximising revenue. Explain why that does not maximise profit. [2 marks]

Revenue is maximised at 80, where its marginal rate is zero.

Profit subtracts a cost increasing at 20 dollars per unit.

Answer: The marginal cost is positive, so the revenue maximum ignores extra production costs.