# Differentiation practice test 5: interpretation and decisions

10 questions | 64 marks | 75 minutes

Original chapter assessment with linked reasoning and applications. Show working, state relevant domains and retain exact values until a numerical answer is requested.

Equivalent correct methods earn the corresponding marks.

**1.** The table gives values and derivatives of two functions at $x=2$. [5 marks]

| $x$ | $u(x)$ | $u'(x)$ | $v(x)$ | $v'(x)$ |
|---|---|---|---|---|
| 2 | 3 | -1 | 5 | 4 |

**(a)** Find $(uv)'(2)$. [2 marks]

Use $u'v+uv'=(-1)(5)+(3)(4)$.

**Answer:** $7$

**(b)** Find $\left(\frac{u}{v}\right)'(2)$. [3 marks]

Use $\frac{(-1)(5)-(3)(4)}{5^2}$.

**Answer:** $-\frac{17}{25}$

**2.** Let $f(x)=\sqrt{9-x^2}$. [6 marks]

**(a)** State the real function domain and find its derivative. [3 marks]

The radicand must be nonnegative.

Use the chain rule and note where the derivative denominator is zero.

**Answer:** $-3\le x\le 3$; $f'=-\frac{x}{\sqrt{9-x^2}}$ for $-3<x<3$

**(b)** Find the tangent at zero and explain why the endpoints are not stationary points. [3 marks]

The derivative is zero at 0 and the height is 3.

Stationary means a defined zero derivative, not merely an endpoint.

**Answer:** $y=3$; the endpoint derivatives are not finite.

**3.** Let $g(x) = x^2 e^{-x}$ for real $x$. In part (b), restrict to $0 \le x \le 4$. [7 marks]

**(a)** Find and classify the stationary points. [4 marks]

$g' = xe^{-x}(2 - x)$.

The derivative changes negative-to-positive at 0 and positive-to-negative at 2.

**Answer:** Local minimum $(0, 0)$; local maximum $(2, 4e^{-2})$.

**(b)** Find the absolute maximum on the interval. [3 marks]

Compare heights at 0, 2 and 4.

The endpoint value $16e^{-4}$ is smaller than $4e^{-2}$.

**Answer:** $4e^{-2}$ at $x = 2$

**4.** For $a > 0$, the identity $a^x = e^{x \ln a}$ is supplied. [5 marks]

**(a)** Derive a formula for the derivative of $a^x$. [3 marks]

Apply the chain rule to $e^{x \ln a}$.

The exponent derivative is the constant $\ln a$.

**Answer:** $(\ln a) a^x$

**(b)** Find the tangent to $y = 3^x$ at zero. [2 marks]

At zero the height is 1 and the gradient is $\ln 3$.

**Answer:** $y = (\ln 3)x + 1$

**5.** Analyse $f(x) = x^5 - 5x$. [6 marks]

**(a)** Find and classify the stationary points. [4 marks]

$f' = 5(x^4 - 1)$ has real roots $\pm 1$.

The derivative changes positive-to-negative at $-1$ and negative-to-positive at 1.

**Answer:** Maximum $(-1, 4)$; minimum $(1, -4)$.

**(b)** Find and justify its inflection point. [2 marks]

$f'' = 20x^3$ changes sign at zero.

**Answer:** $(0, 0)$

**6.** Let $h(x) = \frac{1}{1+x^2}$. [7 marks]

**(a)** Find its first derivative. [2 marks]

Use a negative power and the chain rule.

**Answer:** $-\frac{2x}{(1+x^2)^2}$

**(b)** Show that $h'' = \frac{6x^2-2}{(1+x^2)^3}$, then find the inflection points. [5 marks]

Differentiate $-2x(1+x^2)^{-2}$ with product and chain rules.

The positive denominator means the numerator controls concavity.

It changes sign at both roots of $6x^2 - 2$.

**Answer:** $(-\frac{1}{\sqrt{3}}, \frac{3}{4})$ and $(\frac{1}{\sqrt{3}}, \frac{3}{4})$

**7.** A height is $H(t) = 10 + 4t - t^2$ metres for $0 \le t \le 4$ seconds. [6 marks]

**(a)** Find its average rate from 0 to 2 seconds. [2 marks]

Use $\frac{H(2) - H(0)}{2} = \frac{14 - 10}{2}$.

**Answer:** $2$ m/s

**(b)** Find when its instantaneous rate equals that average. [2 marks]

$H' = 4 - 2t$; set it equal to 2.

**Answer:** $t = 1$ s

**(c)** Find its maximum height. [2 marks]

The derivative is zero at 2 and the quadratic is concave down.

**Answer:** 14 m at 2 s.

**8.** A rectangular enclosure is divided into three equal pens by two internal fences parallel to its width $W$. There is 120 m of fencing for the boundary and the two dividers. [6 marks]

**(a)** Express length $L$ and total area in terms of $W$. [3 marks]

The total fence length is $2L + 4W = 120$.

**Answer:** $L = 60 - 2W$; $A = 60W - 2W^2$, $0 < W < 30$

**(b)** Find dimensions giving greatest total area. [3 marks]

Set $A' = 60 - 4W = 0$.

The second derivative is negative.

**Answer:** $L = 30$ m and $W = 15$ m; total area 450 square metres.

**9.** An open cylindrical container has volume $36\pi$ cubic centimetres. It has a base but no lid. [7 marks]

**(a)** Write its surface area as a function of radius $r$. [3 marks]

The volume constraint gives $h = \frac{36}{r^2}$.

Use base plus curved surface area.

**Answer:** $S = \pi r^2 + \frac{72\pi}{r}$, $r > 0$

**(b)** Find the dimensions minimising material area. [4 marks]

Set $S' = 2\pi r - \frac{72\pi}{r^2} = 0$, obtaining $r^3 = 36$.

Substitute for $h$; the area grows without bound near zero and for large radius.

**Answer:** $r = h = \sqrt[3]{36}$ cm

**10.** A damped displacement model is $s(t) = e^{-t}\cos(2t)$ metres, $t \geq 0$. [9 marks]

**(a)** Find velocity and acceleration. [4 marks]

Use product and chain rules twice.

**Answer:** $v = e^{-t}[-\cos(2t) - 2\sin(2t)]$; $a = e^{-t}[-3\cos(2t) + 4\sin(2t)]$

**(b)** Is the particle speeding up at zero? [2 marks]

At zero, velocity is $-1$ m/s and acceleration is $-3$ m/s squared.

The signs agree.

**Answer:** Yes.

**(c)** Find its first positive rest time to 3 significant figures. [3 marks]

Solve $\tan(2t) = -\frac{1}{2}$.

Choose the first positive angle in quadrant II.

**Answer:** $\frac{\pi - \arctan(\frac{1}{2})}{2} \approx 1.34$ s