Differentiation practice test 4: connected methods

10 questions | 61 marks | 75 minutes

Original chapter assessment with linked reasoning and applications. Show working, state relevant domains and retain exact values until a numerical answer is requested.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = ax^3 + bx^2$, where $f'(1) = 0$ and $f''(1) = 6$. [5 marks]

(a) Find a and b. [3 marks]

The conditions give $3a + 2b = 0$ and $6a + 2b = 6$.

Subtract to find $a = 2$, then $b = -3$.

Answer: $a = 2,\ b = -3$.

(b) Classify the stationary point at $x = 1$. [2 marks]

The supplied second derivative is positive.

Evaluate $f(1) = 2 - 3 = -1$.

Answer: Local minimum $(1, -1)$.

2. Consider $g(x) = (x-1)^2 e^x$. [7 marks]

(a) Differentiate and factorise. [3 marks]

$g' = 2(x-1)e^x + (x-1)^2 e^x$.

Factor the common terms.

Answer: $g' = e^x(x-1)(x+1)$

(b) Find and classify both stationary points. [4 marks]

The exponential is positive.

The derivative signs are positive, negative, positive on the three intervals.

Answer: Maximum $(-1, \frac{4}{e})$; minimum $(1, 0)$

3. Let $h(x)=\frac{\sin x}{1+\cos x}$. [6 marks]

(a) Find its derivative and state the excluded inputs. [4 marks]

The quotient-rule numerator is $\cos x(1+\cos x)+\sin^2 x=1+\cos x$.

Cancel one nonzero denominator factor.

Answer: $h'=\frac{1}{1+\cos x}$; exclude $x=(2k+1)\pi$, integer k.

(b) Find the tangent at zero. [2 marks]

Height is zero and gradient is $\frac{1}{2}$.

Answer: $y=\frac{x}{2}$

4. For $f(x)=\sin(2x)\cos(2x)$, all inputs are radians. [5 marks]

(a) Use the product rule to differentiate. [3 marks]

Differentiate both factors, keeping their inner factor 2.

Answer: $f'=2\cos^2(2x)-2\sin^2(2x)=2\cos(4x)$

(b) Given the equivalent form $f(x)=\frac{1}{2}\sin(4x)$, find $f''(x)$. [2 marks]

Differentiate $2\cos(4x)$ using the chain rule.

Answer: $-8\sin(4x)$

5. A smooth function satisfies $f''(x)=(x-1)^2(x+2)$ for all real x. [7 marks]

(a) Find all inputs at which an inflection could occur. [2 marks]

Solve the second-derivative equation.

Answer: $x=-2, 1$.

(b) Decide which candidate is a true inflection input. [3 marks]

The squared factor has no sign change at 1.

The factor $x+2$ changes sign at -2.

Answer: $x=-2$ only.

(c) Can its inflection height be found from this information alone? [2 marks]

Adding a constant to f preserves its second derivative but changes every height.

Answer: No.

6. A population follows $P(t) = \frac{600}{1+5e^{-0.4t}}$. Time is in days. [8 marks]

(a) Find the initial population and initial rate. [4 marks]

$P(0) = \frac{600}{6} = 100.$

$P' = \frac{1200e^{-0.4t}}{(1+5e^{-0.4t})^2}$; evaluate at zero.

Answer: 100 organisms; $\frac{100}{3}$ organisms per day.

(b) The derivative can also be written as $P' = 0.4P(1 - \frac{P}{600})$. At what population is the growth rate greatest, and what is that rate? [4 marks]

As a function of P, the rate is a downward quadratic.

Its vertex is halfway between zeros 0 and 600.

Evaluate the rate at 300.

Answer: 300 organisms; 60 organisms per day.

7. Let $f(x) = x^3 - 3x$. [5 marks]

(a) Find inputs where the tangent is parallel to $y = 3x + 7$. [2 marks]

Set $3x^2 - 3 = 3.$

Answer: $x = \pm\sqrt{2}$

(b) Find both tangent equations. [3 marks]

The tangent intercept is $f(x) - 3x = x^3 - 6x$.

At $x = \pm\sqrt{2}$, this equals $-4x$.

Answer: $y = 3x - 4\sqrt{2}$ and $y = 3x + 4\sqrt{2}$

8. A rectangle in the first quadrant has one vertex at the origin and its upper-right vertex on $y = 12 - x^2$. Its sides are parallel to the axes. [6 marks]

(a) Form an area function and state a feasible domain. [2 marks]

Width is x and height is $12 - x^2$.

Answer: $A = x(12 - x^2)$ for $0 < x < \sqrt{12}$

(b) Find its maximum area and dimensions. [4 marks]

$A' = 12 - 3x^2 = 0$ gives $x = 2$.

The derivative changes from positive to negative; evaluate the height and area.

Answer: Area 16 square units, width 2 and height 8.

9. A particle has position $s(t) = 1 + \sin(2t)$ metres for $0 \le t \le \pi$ seconds. [6 marks]

(a) Find its reversal times. [3 marks]

$v = 2\cos(2t)$; its sign changes at both zeros.

Answer: $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$

(b) Find total distance and displacement. [3 marks]

Positions at the endpoints and reversal times are $1, 2, 0, 1$.

Add absolute changes: $1 + 2 + 1 = 4$.

Answer: Distance 4 m; displacement 0 m.

10. Total production cost is $C(q) = 0.5q^2 + 200$ dollars for $q > 0$ units. [6 marks]

(a) Form an expression for average cost per unit. [2 marks]

Divide total cost by output.

Answer: $A(q) = 0.5q + \frac{200}{q}$

(b) Minimise average cost, first treating output as continuous, then checking whole units. [4 marks]

The derivative is $0.5 - \frac{200}{q^2}$, zero at 20.

Its sign changes from negative to positive.

The optimum is already an integer.

Answer: 20 per unit at 20 units.