Differentiation practice test 3: representations and models

10 questions | 60 marks | 75 minutes

An original progressively demanding chapter paper. Show analytical working; a scientific calculator may be used for numerical values. Diagrams and data needed for questions are supplied.

Equivalent correct methods earn the corresponding marks.

1. A student claims the derivative of $(x^2+1)\sin x$ is $2x\cos x$. [5 marks]

(a) Explain the error and give the correct derivative. [3 marks]

The product rule adds two contributions rather than multiplying the two derivatives.

Answer: $2x\sin x + (x^2+1)\cos x$

(b) Find the tangent at the origin. [2 marks]

The function is zero at 0 and its derivative there is 1.

Answer: $y = x$

2. Let $f(x) = \frac{1}{(2x-1)^2}$. [5 marks]

(a) Find its first derivative and domain. [3 marks]

Write the function as $(2x-1)^{-2}$ and apply the chain rule.

Answer: $-\frac{4}{(2x-1)^3}$, $x \neq \frac{1}{2}$

(b) Find its second derivative. [2 marks]

Differentiate the power -3 and multiply by inner derivative 2.

Answer: $\frac{24}{(2x-1)^4}$

3. Consider $g(x) = e^{x^2 - 4x}$. [6 marks]

(a) Find and classify its stationary point. [3 marks]

$g' = (2x - 4)e^{x^2 - 4x}$ changes from negative to positive at 2.

Answer: Minimum $(2, e^{-4})$

(b) Show the curve is concave up everywhere. [3 marks]

Differentiate the product; both factors in the result are positive.

Answer: $g'' = [2 + (2x - 4)^2]e^{x^2 - 4x} > 0$

4. Let $h(x) = \sin x + \cos x$ for $0 \le x \le 2\pi$. [6 marks]

(a) Find and classify its stationary points. [4 marks]

$h' = \cos x - \sin x = 0$ gives the two stated inputs.

Use $h'' = -h$ to classify.

Answer: Maximum $(\frac{\pi}{4}, \sqrt{2})$; minimum $(\frac{5\pi}{4}, -\sqrt{2})$

(b) Are these also the absolute extrema on the interval? [2 marks]

Both endpoint values are 1, between the two stationary heights.

Answer: Yes.

5. Analyse $F(x) = x^4 - 4x^3 + 1$. [7 marks]

(a) Find and classify stationary points. [4 marks]

$F' = 4x^2(x - 3)$.

There is no first-derivative sign change at zero, and a negative-to-positive change at 3.

The concavity changes at zero.

Answer: Stationary inflection $(0, 1)$; minimum $(3, -26)$.

(b) Find all inflection points. [3 marks]

$F'' = 12x(x - 2)$ changes sign at 0 and 2.

Evaluate original heights.

Answer: $(0, 1)$ and $(2, -15)$.

6. A model has population $P(t) = 100e^{0.2t}$, with time in hours. Rounded observations are supplied. [6 marks]

t (h)	P (organisms)
0	100
1	122.14
2	149.18
3	182.21

(a) Use the observations at 0 and 2 to estimate the instantaneous rate near 1 hour. [3 marks]

Use the centred difference $\frac{149.18 - 100}{2 - 0}$.

Answer: About 24.59 organisms/h.

(b) Compare with the exact model rate at 1 hour. [3 marks]

Differentiate the given model.

The centred difference is an approximation across a finite interval.

Answer: $20e^{0.2} \approx 24.43$ organisms/h

7. The cubic $f(x) = x^3 + ax^2 + bx + c$ is stationary at $x = -1$ and $x = 1$, and $f(1) = -1$. [6 marks]

(a) Find a and b. [3 marks]

Substitute ± 1 into $3x^2 + 2ax + b = 0$.

Subtract the equations to get $a = 0$, then $b = -3$.

Answer: $a = 0,\ b = -3$.

(b) Find c and the other stationary height. [3 marks]

Use $1 - 3 + c = -1$.

Then evaluate at -1.

Answer: $c = 1;\ f(-1) = 3$.

8. An open box is made by cutting squares of side x from a 20 cm square and folding up the sides. [6 marks]

(a) Give the volume function and feasible domain. [2 marks]

The base is a square of side $20 - 2x$.

Answer: $V = x(20 - 2x)^2,\ 0 < x < 10$

(b) Find the maximum volume. [4 marks]

$V' = 4(3x - 10)(x - 10)$.

The interior candidate is $\frac{10}{3}$; the volume tends to zero at both endpoints.

Answer: $\frac{16000}{27} \approx 592.6\ \text{cm}^3$ at $x = \frac{10}{3}$ cm

9. A point on the line $y = 2x + 1$ is to be as close as possible to the origin. [6 marks]

(a) Write a function for squared distance. [2 marks]

Use $D^2 = x^2 + (2x + 1)^2$.

Answer: $D^2 = 5x^2 + 4x + 1$

(b) Find the point and its distance. [4 marks]

Set the derivative $10x + 4$ to zero.

The squared distance is $\frac{1}{5}$.

Answer: $(-\frac{2}{5}, \frac{1}{5})$; distance $\frac{1}{\sqrt{5}}$

10. A business earns $R(q) = q(50 - q)$ dollars and pays $C(q) = 10q + 100$ dollars for $0 \le q \le 50$ units. [7 marks]

(a) Form the profit function. [2 marks]

Subtract cost from revenue.

Answer: $\Pi(q) = 40q - q^2 - 100$

(b) Find maximum profit and the required output. [3 marks]

$\Pi' = 40 - 2q = 0$ at 20.

The profit function is concave down; evaluate its height.

Answer: 300 dollars at 20 units.

(c) Why is maximising revenue alone not enough? [2 marks]

The objective here is revenue minus cost, not revenue.

Answer: Costs also change with output.