Differentiation practice test 2

10 questions | 61 marks | 75 minutes

An original progressively demanding chapter paper. Show analytical working; a scientific calculator may be used for numerical values. Diagrams and data needed for questions are supplied.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = 2\sqrt{x} + \frac{3}{x}$. [5 marks]

(a) Differentiate and state the derivative domain. [3 marks]

Rewrite using powers and apply the power rule.

Answer: $f' = x^{-\frac{1}{2}} - 3x^{-2}$ for $x > 0$

(b) Find the gradient at $x = 1$. [2 marks]

Substitute into the derivative: $1 - 3$.

Answer: -2

2. Let $g(x) = xe^{-2x}$. [6 marks]

(a) Find the derivative. [3 marks]

Apply product and chain rules.

Answer: $e^{-2x}(1 - 2x)$

(b) Find and classify its stationary point. [3 marks]

The exponential factor is positive.

The derivative changes from positive to negative at $\frac{1}{2}$.

Answer: Maximum $(\frac{1}{2}, \frac{1}{2e})$

3. Let $h(x)=\frac{2x-3}{x^2+1}$. [5 marks]

(a) Find the derivative. [3 marks]

The quotient-rule numerator is $2(x^2+1)-2x(2x-3)$.

Answer: $h'=\frac{-2x^2+6x+2}{(x^2+1)^2}$

(b) Find the tangent at zero. [2 marks]

Height is -3; derivative at zero is 2.

Answer: $y=2x-3$

4. Consider $y=(1-3x^2)^5$. [5 marks]

(a) Differentiate. [2 marks]

The outer derivative is $5(1-3x^2)^4$.

Multiply by $-6x$.

Answer: $-30x(1-3x^2)^4$

(b) Find the second derivative at zero. [3 marks]

Differentiate the product or inspect the linear term of y'.

At zero, the result is -30.

Answer: -30

5. A height model is $H(t)=4+3\cos(\frac{\pi t}{6})$ metres. [6 marks]

(a) Find its rate of change at 3 seconds. [3 marks]

$H'=-(\frac{\pi}{2})\sin(\frac{\pi t}{6})$.

Substitute 3.

Answer: $-\frac{\pi}{2}$ m/s

(b) Find its second derivative at zero. [3 marks]

$H''=-(\frac{\pi^2}{12})\cos(\frac{\pi t}{6})$.

Answer: $-\frac{\pi^2}{12}$ m/s squared

6. A differentiable function has $f'(x) = x^2(x-2)$. [7 marks]

(a) Find the stationary inputs and classify them. [4 marks]

The first derivative is negative on both sides of zero and changes from negative to positive at 2.

Answer: At $x = 0$, neither maximum nor minimum; at $x = 2$, a local minimum.

(b) Find the inputs where concavity changes. [3 marks]

$f'' = x(3x-4)$; its signs are positive, negative, positive across the roots.

Answer: $x = 0$ and $x = \frac{4}{3}$

7. Let $r(x) = \frac{x^2}{x-1}$. [9 marks]

(a) State its vertical and oblique asymptotes. [3 marks]

Divide to obtain $x + 1 + \frac{1}{x-1}$.

Answer: $x = 1$ and $y = x + 1$.

(b) Find and classify the stationary points. [4 marks]

$r' = \frac{x(x-2)}{(x-1)^2}$.

Use the numerator signs on the valid intervals.

Answer: Maximum $(0, 0)$; minimum $(2, 4)$.

(c) Is $x = 1$ an inflection point? [2 marks]

Changing concavity across separate branches does not create a point of inflection.

Answer: No: the original function is undefined there.

8. Temperature is modelled by $T(t) = 30 + 120e^{-0.2t}$ degrees Celsius, with time in minutes. [6 marks]

(a) Find the initial temperature and initial cooling rate. [3 marks]

Evaluate the function and its derivative at zero.

Answer: 150 degrees Celsius; -24 degrees Celsius per minute.

(b) When does temperature reach 75 degrees Celsius? [3 marks]

$45 = 120e^{-0.2t}$, so $e^{-0.2t} = \frac{3}{8}$.

Answer: $5\ln(\frac{8}{3}) \approx 4.90$ min

9. A rectangle has area 64 square metres. [5 marks]

(a) Express its perimeter using one positive side x. [2 marks]

The other side is $\frac{64}{x}$.

Answer: $P = 2x + \frac{128}{x}$, $x > 0$.

(b) Find the least perimeter and justify it. [3 marks]

$P' = 2 - \frac{128}{x^2} = 0$ gives $x = 8$.

The derivative changes from negative to positive.

Answer: 32 m, at an 8 m square.

10. Position is $s(t) = \frac{t^3}{3} - 4t^2 + 15t$ metres, $0 \le t \le 8$ seconds. [7 marks]

(a) Find direction-change times. [3 marks]

$v = (t-3)(t-5)$ changes sign at both inputs.

Answer: 3 s and 5 s.

(b) Find the displacement and total distance. [4 marks]

Positions at times $0, 3, 5, 8$ are $0, 18, \frac{50}{3}, \frac{104}{3}$.

Add the absolute changes for distance.

Answer: $\frac{104}{3}$ m and $\frac{112}{3}$ m