Introduction to calculus - Topic test 5

10 questions | 56 marks | 75 minutes

A harder synthesis paper with exact tangent conditions, parameters, weighted constraints and model evaluation.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = x^3 - 3x$. [4 marks]

(a) Find its average rate on $[-1, 1]$. [2 marks]

$f(-1) = 2, f(1) = -2$; $\frac{-2-2}{1-(-1)} = -2$.

Answer: -2

(b) Compare with its instantaneous rate at 0. [2 marks]

$f' = 3x^2 - 3$, so $f'(0) = -3$.

A secant uses two endpoints; a tangent uses local limiting behaviour.

Answer: -3; the average is not the tangent gradient at 0.

2. A quadratic has form $f(x) = x^2 + bx + c$. [6 marks]

(a) Find the difference quotient at input a and its limit. [3 marks]

$f(a+h) - f(a) = 2ah + h^2 + bh$.

Divide by nonzero h and let h tend to 0.

Answer: $2a + b + h$; limit $2a + b$.

(b) If $f'(3) = 0$ and $f(1) = 5$, find b and c. [3 marks]

$6 + b = 0$ gives $b = -6$. Then $1 - 6 + c = 5$ gives $c = 10$.

Answer: $b = -6$, $c = 10$.

3. Find tangents to $f(x)=x^3-3x$ parallel to $3x-y+2=0$. [6 marks]

(a) Find the possible tangent inputs. [3 marks]

The line gradient is 3. Solve $3x^2-3=3$.

Answer: $x=\pm\sqrt{2}$

(b) Give both exact tangent equations. [3 marks]

At input $a=\pm\sqrt{2}$, $f(a)=-a$ and slope $=3$.

Point-gradient form gives $y+a=3(x-a)$, hence $y=3x-4a$.

Answer: $y=3x-4\sqrt{2}$ and $y=3x+4\sqrt{2}$

4. $f(x)=x^3-3kx$ has a stationary point at $x=2$. [5 marks]

(a) Find k and all stationary coordinates. [3 marks]

$f'=3x^2-3k$; $12-3k=0$ gives $k=4$.

Solve $f'=3x^2-12=0$ and evaluate f.

Answer: $k=4$; $(-2,16)$ and $(2,-16)$.

(b) Classify them. [2 marks]

Derivative signs are $+,-,+$.

Answer: Maximum $(-2,16)$; minimum $(2,-16)$.

5. A model cost is $C(x)=2x^2-16x+50$ for $1\le x\le 6$. [4 marks]

(a) Find its stationary value. [2 marks]

$C'=4x-16=0$ gives 4; $C(4)=32-64+50=18$.

Answer: 18 at $x=4$.

(b) Use all relevant candidates to justify the absolute minimum. [2 marks]

The stationary point is in the domain and both endpoint values are larger.

Answer: 18 is least, compared with $C(1)=36$ and $C(6)=26$.

6. A garden against a wall has two perpendicular sides costing 2 dollars per metre and its remaining parallel side costing 3 dollars per metre. Budget is 120 dollars. Let the perpendicular side be x and parallel side y. [7 marks]

(a) Form the budget constraint and area function. [3 marks]

Both perpendicular sides cost $4x$ in total.

Solve for $y = 40 - \frac{4x}{3}$, then use $A = xy$.

Answer: $4x + 3y = 120$; $A = 40x - \frac{4}{3}x^2$, $0 < x < 30$.

(b) Maximise the area. [4 marks]

$A' = 40 - \frac{8}{3}x = 0$ gives 15.

The derivative changes + to −; $y = 20$ and area $= 300$.

Answer: 300m² at $x = 15$ m, $y = 20$ m.

7. A particle follows $s(t) = t^3 - 6t^2 + 9t$ metres on $0 \le t \le 4$. [6 marks]

(a) When does it move in the negative direction, and what is its speed at $t = 2$? [3 marks]

$v = 3(t-1)(t-3)$ is negative between roots.

$v(2) = -3$, so speed $= 3$.

Answer: For $1 < t < 3$; speed at 2 is 3m/s.

(b) Find its total distance. [3 marks]

Positions at 0,1,3,4 are 0,4,0,4; sum the absolute changes: $4 + 4 + 4$.

Answer: 12m.

8. $f(x) = x^3 - ax + b$ touches the x-axis with a horizontal tangent at $x = 2$. [6 marks]

(a) Find a and b. [3 marks]

$f'(2) = 12 - a = 0$ gives $a = 12$.

$f(2) = 8 - 24 + b = 0$ gives $b = 16$.

Answer: $a = 12$, $b = 16$.

(b) Find the other stationary point and classify both. [3 marks]

$f' = 3(x - 2)(x + 2)$ has signs +,−,+.

$f(-2) = -8 + 24 + 16 = 32$.

Answer: Maximum $(-2, 32)$; minimum $(2, 0)$.

9. Positive numbers a and b satisfy $a + b = 18$. Maximise the product ab^2. [6 marks]

(a) Form an objective and state its domain. [2 marks]

Use $b = 18 - a$ and require both numbers positive.

Answer: $P(a) = a(18 - a)^2$, $0 < a < 18$.

(b) Find the greatest product and the corresponding numbers. [4 marks]

$P' = (18 - a)(18 - 3a)$, so the interior stationary input is 6.

The derivative changes + to −; the boundary limits are 0. $P(6) = 6 \times 12^2 = 864$.

Answer: 864 at $a = 6$, $b = 12$.

10. Consider $M(x) = -x^3 + 12x^2 - 36x + 50$ on $[0, 8]$. [6 marks]

(a) Find all stationary points and classify them. [3 marks]

$M' = -3(x - 2)(x - 6)$ has signs −,+,−.

$M(2) = 18$ and $M(6) = 50$.

Answer: Minimum $(2, 18)$; maximum $(6, 50)$.

(b) Find every input giving the global maximum and explain why a local maximum alone is insufficient. [3 marks]

Endpoint $M(0) = 50$ equals the interior maximum; $M(8) = 18$.

A closed-domain comparison must include boundaries, and an optimum need not be unique.

Answer: Global maximum 50 at $x = 0$ and $x = 6$.