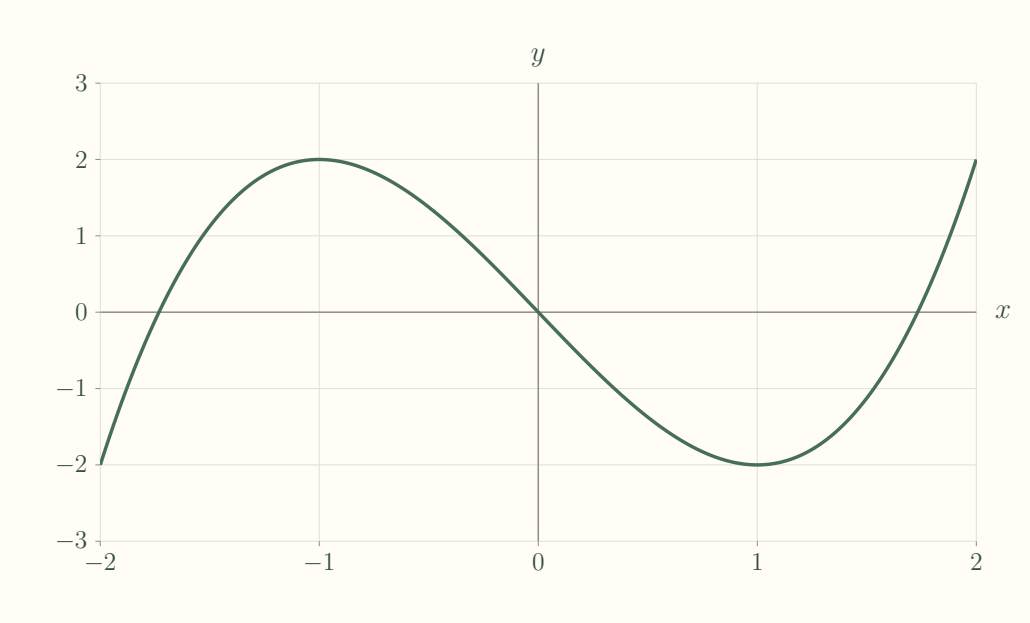
Introduction to calculus - Topic test 4

10 questions | 49 marks | 65 minutes

Analyse different representations, tangent conditions, changes of direction and the validity of solutions.

Equivalent correct methods earn the corresponding marks.

1. The supplied graph represents $f(x) = x^3 - 3x$. [5 marks]



(a) Read the approximate stationary coordinates from the graph. [2 marks]

Identify the horizontal turning points shown.

Answer: Approximately $(-1, 2)$ and $(1, -2)$.

(b) Verify them exactly with a derivative. [3 marks]

$f' = 3x^2 - 3$ is zero at ± 1; signs are +,−,+.

Substitute into f.

Answer: Maximum $(-1, 2)$, minimum $(1, -2)$.

2. Compare $f(x)=x^2$ and $g(x)=2x^2+3$. [4 marks]

(a) Find both derivative functions. [2 marks]

Differentiate each expression term by term.

Answer: $f'=2x,\ g'=4x$.

(b) Based on this comparison, conjecture the derivative of $af(x)+c$ for constants a, c and a differentiable function f. Does this one comparison prove your general rule? [2 marks]

Conjecture that the factor multiplies gradients and the constant disappears: $(af+c)'=af'$. [1 mark]

The calculation checks one function and one choice of constants; it does not prove the general statement. [1 mark]

Answer: Conjecture $(af+c)'=af'$. This example supports the rule but does not prove it for every differentiable function and all constants.

3. $f(x)=x^3+kx$. Its tangent at $x=1$ has gradient 5. [4 marks]

(a) Find k. [2 marks]

$f'=3x^2+k;\ 3+k=5$.

Answer: 2

(b) Find the tangent equation at that point. [2 marks]

With $k=2,\ f(1)=3;\ y-3=5(x-1)$.

Answer: $y=5x-2$

4. For $f(x)=x^3-x$ at the origin, [4 marks]

(a) find the tangent equation. [2 marks]

$f'=3x^2-1$ gives slope -1 at 0.

Answer: $y=-x$

(b) find the normal and verify perpendicularity. [2 marks]

The normal gradient is $\frac{-1}{-1}=1$.

Both pass through the origin and their gradients multiply to -1.

Answer: $y=x$; slope product -1.

5. Analyse $f(x)=x(x-2)^2$. [6 marks]

(a) Find intercepts and stationary coordinates. [4 marks]

$f=x^3-4x^2+4x$; $f'=(3x-2)(x-2)$.

$f(\frac{2}{3})=\frac{32}{27}$ and $f(2)=0$.

Answer: Intercepts $x=0,2$; stationary $(\frac{2}{3},\frac{32}{27})$ and $(2,0)$.

(b) Classify the stationary points. [2 marks]

Derivative signs are $+,-,+$.

Answer: Maximum at $\frac{2}{3}$; minimum at 2.

6. $s(t)=t^3-9t^2+24t$ metres for $0\le t\le 6$. [6 marks]

(a) Find rest times and total distance. [4 marks]

$v=3(t-2)(t-4)$; positions at 0,2,4,6 are 0,20,16,36.

Distance $=20+4+20=44$.

Answer: Rest at 2s and 4s; distance 44m.

(b) At $t=3$, find acceleration and decide whether the particle is at rest. [2 marks]

$a=6t-18$ gives 0; $v(3)=27-54+24=-3$.

Answer: $a=0$ but $v=-3$ m/s, so not at rest.

7. For $C(x)=x^2+2x+3$, [4 marks]

(a) find its unrestricted minimum. [2 marks]

$C'=2x+2=0$ gives -1; $C(-1)=2$.

Answer: 2 at $x=-1$.

(b) find its minimum if only $0\le x\le 4$ is allowed. [2 marks]

The function increases throughout $[0,4]$; evaluate the left endpoint.

Answer: 3 at $x=0$.

8. A rectangular enclosure has a single internal divider parallel to its width. Total fencing is 60m. Let width be x and length y. [6 marks]

(a) Form the area in terms of x alone. [3 marks]

There are three widths and two lengths: $3x + 2y = 60$.

Hence $y = 30 - 1.5x$ and $A = xy$.

Answer: $A = 30x - 1.5x^2$, with $0 < x < 20$.

(b) Find the greatest area and dimensions. [3 marks]

$A' = 30 - 3x = 0$ gives 10; signs + then −.

Then $y = 15$ and $A = 150$.

Answer: 150m²; width 10m and length 15m.

9. $s(t) = t^3 - 6t^2 + 12t$ metres on $0 \le t \le 4$ seconds. [5 marks]

(a) Show the particle is at rest once inside the interval, without turning. [3 marks]

$v = 3(t-2)^2$ is zero only at 2 and positive elsewhere.

Answer: At $t = 2$ only; velocity is nonnegative on both sides.

(b) Find displacement and total distance. [2 marks]

Positions are $s(0) = 0$ and $s(4) = 16$. With no reverse motion, distance equals displacement.

Answer: Both 16m.

10. A student uses $A = x(20 - x)$ for a rectangle and writes $A' = 20 - x$, obtaining $x = 20$. [5 marks]

(a) Correct the derivative and the stationary input. [2 marks]

Expand $A = 20x - x^2$ before differentiating.

Answer: $A' = 20 - 2x$, $x = 10$.

(b) Explain the geometric fault in the proposed $x = 20$ answer. [3 marks]

The physical domain is $0 < x < 20$, and the correct maximum is $A(10) = 100$.

Answer: It gives the other side 0 and area 0; it is not a rectangle of maximum area.