Introduction to calculus - Topic test 3

10 questions | 52 marks | 65 minutes

Select and evaluate constrained models, including non-turning stationary points and boundary optima.

Equivalent correct methods earn the corresponding marks.

1. A production cost is $C(n) = 100 + 3n + 0.02n^2$ dollars, for a continuous model variable n. [5 marks]

(a) Find the average cost increase per unit from $n = 10$ to $n = 30$. [3 marks]

$C(10) = 132$ and $C(30) = 208$; $\frac{208 - 132}{20} = 3.8$.

Answer: 3.80 dollars per unit.

(b) Find $C'(20)$ and compare the two rates. [2 marks]

$C' = 3 + 0.04n$; at 20 this is 3.8.

Equality here does not make every secant equal every tangent.

Answer: 3.80 dollars per unit; equal in this quadratic example.

2. Let $f(x) = (x - 3)^2$. [4 marks]

(a) Expand f and find its derivative. [2 marks]

Expand first, then apply the polynomial rule.

Answer: $f = x^2 - 6x + 9$, $f' = 2x - 6$.

(b) Explain why its tangent is horizontal at its minimum. [2 marks]

$2x - 6$ is negative before 3 and positive after 3.

Answer: At $x = 3$, $f' = 0$ and changes − to +.

3. A tangent to $y = x^2 + 4$ touches at input a and passes through the origin. [6 marks]

(a) Express its equation in terms of a. [3 marks]

The point is $(a, a^2 + 4)$ and gradient $2a$.

Expand $y - (a^2 + 4) = 2a(x - a)$.

Answer: $y = 2ax + 4 - a^2$

(b) Find both possible tangents. [3 marks]

Passing through $(0, 0)$ requires $4 - a^2 = 0$, hence $a = \pm 2$.

Answer: $y = 4x$ and $y = -4x$.

4. Consider $f(x) = x^3 - 6x^2 + 12x$ on $[0, 4]$. [5 marks]

(a) Find its stationary point and decide whether it is a turn. [3 marks]

$f' = 3(x - 2)^2$; it is positive on both sides of 2.

$f(2) = 8$.

Answer: $(2, 8)$, stationary but not turning.

(b) Find its greatest and least values on the interval. [2 marks]

The function is increasing across the domain; evaluate endpoints.

Answer: Least 0 at $x = 0$; greatest 16 at $x = 4$.

5. A restricted cost model is $C(x) = x^2 - 8x + 30$ for $5 \le x \le 10$. [4 marks]

(a) Find the unrestricted stationary input and assess its relevance. [2 marks]

$C' = 2x - 8 = 0$ gives 4.

Answer: $x = 4$ is outside the permitted interval.

(b) Find the permitted minimum. [2 marks]

C' is positive throughout $[5, 10]$, so the smallest value is $C(5) = 25 - 40 + 30 = 15$.

Answer: 15 at $x = 5$.

6. Squares of side x cm are cut from the corners of an 18cm square, which is folded into an open box. [6 marks]

(a) Write its volume and physical domain. [2 marks]

Height is x and both base dimensions are $18-2x$.

Answer: $V=x(18-2x)^2$, $0<x<9$.

(b) Find and justify its maximum volume. [4 marks]

$V'=12(x-3)(x-9)$; the interior root is 3.

V' changes + to − at 3; $V(3)=3\times12^2=432$.

Answer: 432cm³ at $x=3$ cm.

7. A profit model is $P(x)=-x^3+9x^2-15x$ on $[0,6]$. [6 marks]

(a) Find its stationary coordinates and types. [4 marks]

$P'=-3(x-1)(x-5)$ gives signs −,+,−.

Evaluate $P(1)=-7$ and $P(5)=25$.

Answer: Minimum $(1,-7)$; maximum $(5,25)$.

(b) Find the global maximum on the interval. [2 marks]

Endpoint values $P(0)=0$ and $P(6)=18$ are below 25.

Answer: 25 at $x=5$.

8. A vertical height model is $h(t) = -5t^2 + 20t + 1$ metres after t seconds, from launch until the object first reaches ground level. Take upwards as positive. [5 marks]

(a) Find the maximum height and when it is reached. [3 marks]

$v = h' = -10t + 20 = 0$ at 2; $h(2) = 21$.

Velocity changes + to −.

Answer: 21m at 2s.

(b) Find acceleration and interpret its sign. [2 marks]

$a = v' = -10$.

The sign describes acceleration, not necessarily downward velocity at every instant.

Answer: −10m/s², directed downward when up is positive.

9. The price per item is $p = 30 - 0.5x$ dollars and cost is $C = 5x + 20$, for $0 \le x \le 50$. [6 marks]

(a) Form the profit function. [2 marks]

Revenue is $xp = 30x - 0.5x^2$; subtract the cost.

Answer: $P = -0.5x^2 + 25x - 20$

(b) Find the profit-maximising quantity, price and profit. [4 marks]

$P' = 25 - x = 0$ gives 25 with a maximum.

$p(25) = 17.5$ and $P(25) = -312.5 + 625 - 20 = 292.5$.

Answer: $x = 25$; price 17.50 dollars; profit 292.50 dollars.

10. A derivative is $f'(x) = (x+1)^2(x-2)$. [5 marks]

(a) Determine its sign intervals. [3 marks]

The squared factor is nonnegative, so the sign away from -1 follows $x - 2$.

Answer: Negative for $x < 2$ except zero at -1; positive for $x > 2$.

(b) Classify the stationary inputs. [2 marks]

At -1 there is no sign change; at 2 it changes − to +.

Answer: $x = -1$ is non-turning; $x = 2$ is a local minimum.