Introduction to calculus - Topic test 2

10 questions | 52 marks | 65 minutes

Move between numerical data, derivative information and polynomial graphs, then interpret models.

Equivalent correct methods earn the corresponding marks.

1. A position table gives $s(0)=3$, $s(1)=5$, $s(2)=11$ and $s(3)=21$ metres. [5 marks]

Position measurements

t (s)	0	1	2	3
s (m)	3	5	11	21

(a) Find the average velocities on $[0,1]$, $[1,2]$ and $[2,3]$. [3 marks]

Subtract consecutive positions; each interval is one second.

Answer: 2, 6 and 10m/s.

(b) Use $[1,2]$ to estimate velocity near 1.5s. Is it necessarily exact? [2 marks]

A secant slope approximates a local rate; a function or limiting evidence is needed for exactness.

Answer: About 6m/s; not necessarily exact.

2. Use $f(x)=x^3$ at $x=1$. [4 marks]

(a) Simplify its difference quotient. [3 marks]

Expand $(1+h)^3-1=3h+3h^2+h^3$ and divide by $h\neq 0$.

Answer: $3+3h+h^2$

(b) Find the instantaneous gradient. [1 mark]

Let h tend to 0 in the simplified quotient.

Answer: 3

3. A quadratic is $f(x)=ax^2+bx+c$, with $f'(0)=-4$, $f'(2)=8$ and $f(0)=5$. [4 marks]

(a) Find a and b. [3 marks]

$f'=2ax+b$; at 0, $b=-4$; at 2, $4a-4=8$, giving $a=3$.

Answer: $a=3$, $b=-4$.

(b) Write the full function. [1 mark]

The original value at 0 gives $c=5$.

Answer: $f(x)=3x^2-4x+5$

4. For $y=x^3$, [6 marks]

(a) find the normal at $x=1$. [3 marks]

The point is $(1,1)$; tangent gradient is 3; normal gradient is $-\frac{1}{3}$.

Answer: $y-1=-\frac{1}{3}(x-1)$

(b) find all points where the tangent gradient is 3. [3 marks]

$3x^2=3$ gives $x=\pm1$; substitute into x^3.

Answer: $(-1,-1)$ and $(1,1)$.

5. Compare $f(x)=x^3$ and $g(x)=-x^4$ at $x=0$. [5 marks]

(a) Show that each has a stationary point there. [2 marks]

$f'=3x^2$ and $g'=-4x^3$.

Answer: Both derivatives are 0 at 0.

(b) Classify each using signs. [3 marks]

For f, slopes are positive on both sides. For g, slopes change positive to negative.

Answer: f: neither; g: local maximum.

6. Analyse $f(x) = x^3 - 6x^2 + 9x$. [7 marks]

(a) Find all intercepts and stationary coordinates. [4 marks]

$f = x(x-3)^2$; $f' = 3(x-1)(x-3)$.

Evaluate f at 1 and 3.

Answer: Intercepts $(0,0), (3,0)$; stationary $(1,4), (3,0)$.

(b) Give monotonic intervals and stationary types. [3 marks]

Derivative signs are +,−,+.

Answer: Increasing $x < 1$ or $x > 3$, decreasing $1 < x < 3$; maximum $(1,4)$, minimum $(3,0)$.

7. $s(t) = t^3 - 3t^2$ metres on $0 \le t \le 4$ seconds. [6 marks]

(a) Find rest times and the interior change of direction. [3 marks]

$v = 3t(t-2)$. It is negative on $(0,2)$, positive on $(2,4)$.

Answer: At rest at 0s and 2s; interior turn at 2s.

(b) Find displacement and total distance. [3 marks]

Positions at 0,2,4 are $0, -4, 16$; distance $4 + 20 = 24$.

Answer: Displacement 16m; distance 24m.

8. A garden against a wall has 48m of fencing for the other three sides. [5 marks]

(a) Form its area in terms of the perpendicular side x. [2 marks]

The parallel side is $48 - 2x$.

Answer: $A = x(48 - 2x)$ for $0 < x < 24$.

(b) Find the greatest area. [3 marks]

$A' = 48 - 4x$ is 0 at 12, changing + to −; $A(12) = 288$.

Answer: 288m²; sides 12m,12m,24m.

9. Cost is $C(x) = 0.25x^2 - 5x + 90$ on $0 \le x \le 30$. [5 marks]

(a) Find and classify its stationary point. [3 marks]

$C' = 0.5x - 5 = 0$ gives 10; signs − to +.

Answer: Minimum at $x = 10$, cost 65.

(b) Check that it is the global minimum on this domain. [2 marks]

$C(0) = 90, C(30) = 225 - 150 + 90 = 165.$

Both exceed 65.

Answer: Yes: endpoint costs are 90 and 165.

10. A quadratic has derivative $f'(x) = 2x - 4$ and $f(1) = 7$. [5 marks]

(a) Find f by comparing quadratic coefficients. [3 marks]

Write $f = ax^2 + bx + c$. Its derivative gives $a = 1, b = -4$; $f(1) = 7$ gives $c = 10$.

Answer: $f(x) = x^2 - 4x + 10$

(b) Find its minimum value. [2 marks]

$f' = 0$ at 2, with − to +; $f(2) = 4 - 8 + 10 = 6$.

Answer: 6 at $x = 2$.