Introduction to calculus - Topic test 1

10 questions | 48 marks | 65 minutes

A balanced first paper: rates, derivative rules, tangent lines, graph behaviour, optimisation and motion.

Equivalent correct methods earn the corresponding marks.

1. A position model is $s(t) = t^2 + 1$ metres. [4 marks]

(a) Find the average velocity from 1s to 3s. [2 marks]

$s(1) = 2, s(3) = 10$; $\frac{10-2}{2} = 4$.

Answer: 4m/s

(b) Compare this with the average from 0s to 2s and interpret the difference. [2 marks]

$s(0) = 1, s(2) = 5$; $\frac{5-1}{2} = 2$.

The model gains position faster over the later interval.

Answer: 2m/s; later average velocity is greater.

2. For $f(x) = x^2$, investigate the gradient at $x = 2$. [4 marks]

(a) Simplify $\frac{f(2+h)-f(2)}{h}$ for $h \neq 0$. [2 marks]

$(2+h)^2 - 4 = 4h + h^2$; divide by h.

Answer: $4 + h$

(b) State the limiting gradient and explain why $h = 0$ is not substituted first. [2 marks]

The simplified expression tends to 4; division by zero is undefined.

Answer: 4; the initial quotient would be $\frac{0}{0}$.

3. Let $f(x) = 2x^3 - 3x^2 + 4$. [4 marks]

(a) Find its derivative function. [2 marks]

Apply the power rule term by term.

Answer: $f'(x) = 6x^2 - 6x$

(b) Find $f'(2)$ and distinguish it from $f(2)$. [2 marks]

$f'(2) = 24 - 12 = 12$; $f(2) = 16 - 12 + 4 = 8$.

One is a slope and the other a height.

Answer: 12 and 8, respectively.

4. Consider $y = x^2 - 2x + 3$ at $x = 2$. [5 marks]

(a) Find the tangent equation. [3 marks]

The point is $(2, 3)$ and derivative $2x - 2$ gives gradient 2.

Use $y - 3 = 2(x - 2)$.

Answer: $y = 2x - 1$

(b) Find the normal equation. [2 marks]

Normal gradient is $-\frac{1}{2}$ through $(2, 3)$.

Answer: $y = -\frac{1}{2}x + 4$

5. For $f(x) = x^3 - 3x$, [6 marks]

(a) find intervals of increase and decrease. [3 marks]

$f' = 3(x-1)(x+1)$ has signs +,−,+.

Answer: Increasing $x < -1$ or $x > 1$; decreasing $-1 < x < 1$.

(b) find and classify both stationary points. [3 marks]

Evaluate f at -1 and 1 and use the sign changes.

Answer: Maximum $(-1, 2)$, minimum $(1, -2)$.

6. You are given $g'(x) = -(x+2)(x-1)$. [5 marks]

(a) Construct a sign chart. [2 marks]

Test values $-3, 0, 2$.

Answer: − for $x < -2$; + for $-2 < x < 1$; − for $x > 1$.

(b) State the stationary types and explain what is needed to find their heights. [3 marks]

Signs − to + and + to − determine types, but a derivative does not fix vertical position.

Answer: Minimum at -2, maximum at 1; the original g or a height condition is needed.

7. A rectangle has perimeter 32m. Let its width be x metres. [5 marks]

(a) Form an area function and state its physical domain. [2 marks]

The other side is $16 - x$; both lengths must be positive.

Answer: $A = x(16 - x)$; $0 < x < 16$

(b) Find and justify its maximum area. [3 marks]

$A' = 16 - 2x = 0$ at 8; signs + then −.

$A(8) = 64$.

Answer: 64m² with both sides 8m.

8. Maximise $f(x) = -x^2 + 6x$ over $0 \le x \le 2$. [4 marks]

(a) Find its stationary input and say whether it is allowed. [2 marks]

$f' = 6 - 2x = 0$ gives 3.

Answer: $x = 3$, outside the domain.

(b) Determine the maximum on the given interval. [2 marks]

The derivative is positive throughout the interval; $f(2) = 8$.

Answer: 8 at $x = 2$.

9. $s(t) = t^2 - 4t$ metres on $0 \le t \le 5$ seconds. [6 marks]

(a) Find the rest time and the direction on either side. [3 marks]

$v = 2t - 4$ changes sign at 2.

Answer: Rest at 2s; negative before, positive after.

(b) Find displacement and total distance. [3 marks]

Positions at 0,2,5 are $0,\ -4, 5$; distance $4 + 9 = 13$.

Answer: Displacement 5m; distance 13m.

10. Revenue is $R(x) = x(20 - x)$ dollars and cost is $C(x) = 4x + 12$, for $0 \le x \le 15$. [5 marks]

(a) Form the profit function. [2 marks]

Profit is revenue minus cost.

Answer: $P(x) = -x^2 + 16x - 12$

(b) Find the greatest modelled profit and the corresponding x. [3 marks]

$P' = 16 - 2x = 0$ at 8; the downward parabola has a maximum.

$P(8) = -64 + 128 - 12 = 52$.

Answer: 52 dollars at $x = 8$.