Marking guide | Exponentials and logarithms

Exponential and logarithmic functions - Applications test

10 questions | 53 marks | 75 minutes

Apply exact algebra and logarithmic models to medicine, finance, calibrated scales and the reliability of forecasts.

Equivalent correct methods earn the corresponding marks.

1. Calculate exact values and handle signs carefully. [4 marks]

(a) Evaluate $125^{\frac{2}{3}}$ and $16^{-\frac{1}{2}}$. [2 marks]

The cube root of 125 is 5; square it. The negative half-power is $\frac{1}{\sqrt{16}} = \frac{1}{4}$.

Answer: 25 and $\frac{1}{4}$.

(b) Simplify $\sqrt{x^2}$ for $x = -7$ and explain why it is not x. [2 marks]

$\sqrt{(-7)^2} = \sqrt{49} = 7 = |x|$.

Answer: 7; the principal square root is non-negative.

2. Work exactly before rounding. [4 marks]

(a) Rationalise $\frac{6}{\sqrt{7}-1}$. [2 marks]

Multiply by the conjugate: $\frac{6(\sqrt{7}+1)}{7-1} = \sqrt{7}+1$.

Answer: $\sqrt{7}+1$

(b) Find its value to three decimal places. [2 marks]

Evaluate the exact expression $\sqrt{7}+1$ once, then round.

Answer: 3.646

3. Two laboratory quantities, in milligrams, follow $A(t)=120(1.25)^t$ and $B(t)=300(0.9)^t$, where t is hours. Assume both rules are valid for $0\le t\le 5$. [5 marks]

(a) Form an equation for the time when the amounts are equal. [2 marks]

Set $120(1.25)^t=300(0.9)^t$ and divide by the positive factor $120(0.9)^t$.

Answer: $(\frac{1.25}{0.9})^t=2.5$

(b) Solve the equation and identify which amount is larger at $t=0$. [3 marks]

$t=\frac{\log 2.5}{\log(\frac{1.25}{0.9})}$.

At zero, $A=120$ and $B=300$.

Answer: $t\approx 2.789$; B is larger initially.

4. A medicine model is $M(t)=48(0.5)^{\frac{t}{4}}$ mg, where t is hours after one dose. Assume no further doses. [5 marks]

(a) State its half-life and calculate $M(10)$. [2 marks]

The exponent counts four-hour periods. At 10 hours it is 2.5, so $M(10)=48(0.5)^{2.5}$.

Answer: 4 hours; 8.485 mg.

(b) Find all times $t\ge 0$ for which the model predicts less than 5 mg. [3 marks]

Solve $48(0.5)^{\frac{t}{4}}<5$. A negative logarithm reverses the inequality when dividing.

Answer: $t>\frac{4\log(\frac{5}{48})}{\log 0.5}\approx 13.052$ hours.

5. A transformed exponential has horizontal asymptote $y=2$ and passes through $(0,5)$ and $(1,11)$. Its rule is $y=k+Ab^x$. [5 marks]

(a) Find k, A and b. [3 marks]

The asymptote gives $k=2$. At zero, $2+A=5$. At one, $2+3b=11$.

Answer: $k=2,\ A=3,\ b=3$.

(b) Predict its value at $x=2$ and state its range. [2 marks]

$2+3(3^2)=29$; the exponential contribution is positive.

Answer: $y=29$; range $y>2$.

6. A scale is defined by $p = -\log C$, where C is a positive concentration in a fixed unit. [6 marks]

(a) Find C if $p = 4.5$. [3 marks]

From $\log C = -4.5$, use the matching base-ten exponential.

Answer: $C = 10^{-4.5} \approx 3.1623 \times 10^{-5}$

(b) If p increases from 4.5 to 6.5, by what factor does C change? [3 marks]

The ratio is $\frac{10^{-6.5}}{10^{-4.5}} = 10^{-2}$.

Answer: It is multiplied by 0.01; it becomes one hundredth as large.

7. A sound-level model is $L = 10\log(\frac{I}{I_0})$, where I and the fixed reference I_0 are positive intensities. [5 marks]

(a) Find the increase in L when I is multiplied by 100. [2 marks]

The change is $10\log 100 = 20$.

Answer: 20 units.

(b) A level rises by 3 units. Find the intensity multiplier to three decimals. [3 marks]

$3 = 10\log(\frac{I_2}{I_1})$, so the multiplier is $10^{0.3}$.

Answer: 1.995

8. An initial 6000 dollars earns a nominal 4.8% per annum compounded monthly, with no fees or extra payments. [6 marks]

(a) Write the balance after m complete months and calculate it after 18 months. [3 marks]

The monthly rate is $\frac{0.048}{12} = 0.004$; substitute $m = 18$.

Answer: $V(m) = 6000(1.004)^m$; 6447.01 dollars.

(b) Find the first complete month in which the balance exceeds 7000 dollars. [3 marks]

$m > \frac{\log(\frac{7000}{6000})}{\log 1.004}$; choose the first larger whole month.

Answer: 39 months.

9. Solve $2\log(x-1)=\log(3x+1)$. [5 marks]

(a) State the original domain and obtain a polynomial equation. [3 marks]

$x-1>0$ and $3x+1>0$ together give $x>1$.

$(x-1)^2=3x+1$ expands to $x^2-5x=0$.

Answer: $x>1$; $x^2-5x=0$.

(b) Find the valid root and verify it. [2 marks]

The roots are 0 and 5; only 5 satisfies $x>1$.

At 5, $2\log 4=\log 16$.

Answer: $x=5$.

10. A researcher fits the first three equally spaced observations below with a pure exponential model. [8 marks]

Observed quantity

Time t	0	1	2	3
Observed value	50	75	112.5	160

(a) Find that model and its prediction at $t=3$. [3 marks]

The first three ratios are both 1.5; 50 is the time-zero value.

$50(1.5)^3=168.75$.

Answer: $y=50(1.5)^t$; prediction 168.75.

(b) The actual $t=3$ observation is 160. Find the absolute error and the percentage error relative to the actual value. [3 marks]

$|168.75-160|=8.75$.

$100(\frac{8.75}{160})=5.46875\%$.

Answer: Absolute error 8.75; percentage error approximately 5.47%.

(c) Explain why this fourth observation matters before forecasting much further. [2 marks]

The later process may differ from the original fitted trend; inspect further data and relevant constraints before extrapolating.

Answer: It shows that the fixed multiplier does not continue exactly.