Exponential and logarithmic functions - Practice test 3

10 questions | 50 marks | 75 minutes

Develop and compare models with different time periods, background levels and practical thresholds.

Equivalent correct methods earn the corresponding marks.

1. A function is $f(x) = 5(4^x)$. [5 marks]

(a) Find $f(0)$ and the range. [2 marks]

$4^0 = 1$ and $4^x > 0$ for all real x.

Answer: $f(0) = 5$; range $y > 0$.

(b) Make x the subject of $y = f(x)$, and state the domain of the resulting inverse rule. [3 marks]

Divide by 5, then apply a common logarithm and its power law.

Answer: $x = \frac{\log(\frac{y}{5})}{\log 4}$ for $y > 0$.

2. For this question x is positive. [4 marks]

(a) Simplify $(81x^{12})^{\frac{1}{4}}$. [2 marks]

$81^{\frac{1}{4}} = 3$ and $\frac{12}{4} = 3$.

Answer: $3x^3$

(b) Simplify $\frac{\sqrt{45x^2}}{\sqrt{5}}$. [2 marks]

Combine the roots to get $\sqrt{9x^2}$; $x > 0$ gives $3x$.

Answer: $3x$

3. A cooling model is $T(t) = 18 + 62(0.85)^t$, where T is in degrees Celsius and t is in minutes. [6 marks]

(a) Find the initial temperature and explain the meaning of 18. [2 marks]

$T(0) = 18 + 62 = 80$; the exponential term tends to zero.

Answer: 80 degrees Celsius; 18 is the limiting/background temperature.

(b) For which times does the model predict a temperature below 25 degrees Celsius? [4 marks]

$18 + 62(0.85)^t < 25$ gives $(0.85)^t < \frac{7}{62}$.

Dividing its log inequality by $\log 0.85 < 0$ reverses the sign.

Answer: $t > \frac{\log(\frac{7}{62})}{\log 0.85} \approx 13.42$ minutes.

4. A culture starts with 300 cells and doubles every 30 minutes. Assume the model remains useful for the period considered. [5 marks]

(a) Write a model with t measured in hours. [2 marks]

An hour contains two doubling periods; the hourly factor is 4.

Answer: $N(t) = 300(4^t)$.

(b) Predict its size after 75 minutes. [3 marks]

75 minutes is 2.5 doubling periods; $300(2^{2.5}) \approx 1697.06$.

Answer: Approximately 1697 cells.

5. Account A pays a nominal 5.8% per annum compounded monthly. Account B pays 5.9% compounded annually. Both have no fees. [5 marks]

(a) Find the effective annual rate for A. [3 marks]

Use $100[(1+\frac{0.058}{12})^{12}-1]$.

Answer: 5.9567%.

(b) For the same initial balance, which account has the larger balance after one year? Explain. [2 marks]

Compare its effective annual rate, approximately 5.96%, with 5.9%.

The nominal rates alone are not the fair comparison.

Answer: Account A.

6. Solve, retaining only valid real solutions. [5 marks]

(a) $\log_2(x-1)=3$. [2 marks]

$x-1=8$; hence $x=9$, which satisfies $x>1$.

Answer: $x=9$.

(b) $\log(x-1)+\log(x+1)=\log 24$. [3 marks]

Domain $x>1$; $x^2-1=24$ gives $x=5$ or -5; only 5 is valid.

Answer: $x=5$.

7. A radioactive sample has 80 g initially and 10 g after 15 days. Model its decay using a fixed half-life. [5 marks]

(a) Find its half-life. [2 marks]

$\frac{10}{80}=\frac{1}{8}$ is three halvings; $\frac{15}{3}=5$ days.

Answer: 5 days.

(b) Find when 1 g remains, to two decimals. [3 marks]

$80(\frac{1}{2})^{\frac{t}{5}}=1$; solve using common logs.

Answer: 31.61 days.

8. An observer records $y = 0$ at $t = 0$ and $y = 5$ at $t = 1$. [5 marks]

(a) Can $y = Ab^t$ with $b > 0$ fit both points? Explain. [3 marks]

At $t = 0$, $A = 0$. The formula would then be zero for every t.

Answer: No.

(b) Give one alternative type of model that can fit both points, with an example. [2 marks]

Substitute $t = 0$ and $t = 1$ to check both values.

Answer: A linear model, $y = 5t$.

9. A quantity is modelled by $Q(t) = 20(2^t)$ and is checked only after completed whole hours. [4 marks]

(a) In which hour does it first reach at least 640? [2 marks]

$20(2^5) = 640$. The phrase at least includes equality.

Answer: Hour 5.

(b) In which hour does it first exceed 640? [2 marks]

The continuous condition is $t > 5$; the next whole hour is 6.

Answer: Hour 6.

10. Over a short interval, the radius of a growing sphere follows $r(t) = 10(1.03)^t$ centimetres. Use $V = \frac{4\pi r^3}{3}$. [6 marks]

(a) Write an exponential formula for its volume. [3 marks]

Substitute the radius and cube the complete factor $10(1.03)^t$.

Answer: $V(t) = \frac{4000\pi}{3}(1.03)^{3t}$ cubic centimetres.

(b) Find its percentage volume increase per time unit. [3 marks]

The volume multiplier is 1.03^3, not 1.03.

Answer: 9.2727%.