Matrix models - Applications test 6

10 questions | 59 marks | 70 minutes

Linked models, inference and justified recommendations. Network matrices use rows=start and columns=destination; transition matrices use columns=current and rows=next unless stated otherwise.

Equivalent correct methods earn the corresponding marks.

1. A closed two-state matrix has rows $\begin{pmatrix} 0.9 & b \\ 0.1 & 1-b \end{pmatrix}$. A current state $(60, 40)$ is predicted to become $(62, 38)$. [6 marks]

(a) Find b and check the resulting matrix. [3 marks]

$0.9(60) + 40b = 62$, so $b = \frac{8}{40} = 0.2$.

Answer: $b = 0.2$; rows $\begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$ are column-stochastic.

(b) Find steady counts for a total of 300. [3 marks]

Balance gives $0.1a = 0.2c$, so ratio 2:1.

Answer: $(200, 100)$.

2. A service-count adjacency matrix has rows $\begin{pmatrix} 0 & k & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$, ordered A,B,C, with k a nonnegative integer. [6 marks]

(a) If there are six two-step walks A to C, find k. [3 marks]

The (A, C) entry of the square is $2k$.

Solve $2k = 6$.

Answer: $k = 3$.

(b) Find the number of three-step A-to-C walks and explain the route. [3 marks]

The A row of the square is $(1, 0, 6)$, whose dot product with column C is 1.

Answer: 1: A→C→A→C.

3. Three towns form the directed cycle A→B→C→A. [5 marks]

(a) Find which of $A_{AB}, (A^2)_{AB}, (A^3)_{AB}, (A^4)_{AB}$ are nonzero. [3 marks]

A returns to the same phase every three steps.

Answer: The first and fourth are 1; the other two are 0.

(b) Explain why a route-planning app must distinguish reachability from a required exact number of transfers. [2 marks]

For example one or four moves works; two or three does not.

Answer: B is reachable from A, but only certain exact lengths occur in this network.

4. A ranking rule counts direct wins and two-step wins. Teams A,B,C form the cycle A beats B, B beats C, C beats A. [5 marks]

(a) Show that the rule cannot select a unique winner if all direct wins get weight 1 and all two-step wins the same weight w. [3 marks]

Each team has one direct and one two-step relation.

Answer: Every team scores $1 + w$.

(b) Give two pieces of additional evidence that could improve a ranking model without merely inventing a tie-break. [2 marks]

The extra evidence and weighting rule must be transparent.

Answer: Examples: match margins, repeat-match results or independently specified strength-of-opposition information.

5. A 100-member population uses transition rows $\begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix}$. The next state is $(60, 40)$. [6 marks]

(a) Recover the current state. [4 marks]

Let current A be a. Then $0.9a + 0.4(100 - a) = 60$.

Thus $0.5a = 20$, so $a = 40$.

Answer: $(40, 60)$.

(b) Verify both destination counts. [2 marks]

Both equations agree with the observed forecast.

Answer: A:$36 + 24 = 60$; B:$4 + 36 = 40$.

6. Start at $(100, 0)$. Let T have rows $\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$, and B rows $\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$. [7 marks]

(a) Apply policies T, then B, then T. Write the product and final state. [4 marks]

The states are $(80, 20)$, then $(60, 40)$, then $(60, 40)$.

Answer: $TBTx_0 = (60, 40)$.

(b) Compare the final state for policies B, then T, then B. [3 marks]

The states are $(70, 30)$, then $(65, 35)$, then $(52.5, 47.5)$.

Answer: $BTBx_0 = (52.5, 47.5)$.

7. A three-state matrix has rows $\begin{pmatrix} 0.5 & a & 0 \\ 0.5 & 1-2a & 0.5 \\ 0 & a & 0.5 \end{pmatrix}$. [5 marks]

(a) Find the allowable range of a for a transition matrix. [2 marks]

Entries a must be nonnegative and $1-2a$ must be nonnegative. Columns already sum to 1.

Answer: $0 \le a \le 0.5$.

(b) Find a if $(25, 50, 25)$ is steady. [3 marks]

First-state balance gives $25 = 0.5(25) + 50a$.

Solve $50a = 12.5$ and verify the other states.

Answer: $a = 0.25$.

8. A two-state policy keeps 85% of current A in A. The proportion of B moving to A is b. [7 marks]

(a) Choose b so the steady A proportion is 0.7. [4 marks]

Steady balance is $0.15(0.7) = b(0.3)$.

Thus $b = \frac{0.105}{0.3} = 0.35$.

Answer: $b = 0.35$.

(b) If b rises to 0.4, find the new steady A proportion and interpret. [3 marks]

A larger incoming B-to-A proportion raises the steady share of A, under unchanged assumptions.

Answer: $\frac{0.4}{0.15 + 0.4} = \frac{8}{11} \approx 0.7273$.

9. A model uses rows $\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$, initially $(200, 80)$. The observed next state is $(180, 100)$. [5 marks]

(a) Find the one-period prediction and maximum absolute error. [3 marks]

A: $160 + 24 = 184$; B: $40 + 56 = 96$.

Answer: Prediction $(184, 96)$; maximum absolute error 4.

(b) The model's two-period prediction is exactly the later observation $(176, 104)$. Does this validate the model permanently? [2 marks]

Continue checking independent data and assumptions.

Answer: No; an exact match at one later time does not establish future constant behaviour.

10. Options F and S use transition rows $\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$ and $\begin{pmatrix} 0.98 & 0.03 \\ 0.02 & 0.97 \end{pmatrix}$, respectively. Start at $(100, 0)$. Per-period service cost is AUD 5 for A and AUD 1 for B. [7 marks]

(a) Compare expected costs using the distribution after three periods. [4 marks]

A counts are 65 and $60 + 40(0.95)^3 = 94.295$.

Costs are $5a + (100 - a) = 100 + 4a$.

Answer: F: AUD 360; S: AUD 477.18.

(b) Compare steady-state costs and explain why the preferred short-term option need not follow from equilibrium alone. [3 marks]

Both steady proportions are $(0.6, 0.4)$.

A fixed long-run split does not specify the speed of approach or transition-period cost.

Answer: Both steady costs are AUD 340, but their finite-period distributions differ.