Matrix models - Practice test 4

10 questions | 51 marks | 70 minutes

Structure, exceptional behaviour and careful interpretation. Network matrices use rows=start and columns=destination; transition matrices use columns=current and rows=next unless stated otherwise.

Equivalent correct methods earn the corresponding marks.

1. Let A have rows $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$, and let I be the identity matrix. [5 marks]

(a) Find AI and IA. [2 marks]

The identity leaves each row/column combination unchanged.

Answer: Both equal A.

(b) Calculate A^2 and explain why it is not $2A$. [3 marks]

For A^2, row 1 is $1(1)+2(0)=1$, $1(2)+2(1)=4$; row 2 is $0(1)+1(0)=0$, $0(2)+1(1)=1$.

Therefore A^2 has rows $\begin{pmatrix}1 & 4\\0 & 1\end{pmatrix}$.

Doubling entries instead gives $2A$ with rows $\begin{pmatrix}2 & 4\\0 & 2\end{pmatrix}$, a different operation.

Answer: A^2 has rows $\begin{pmatrix}1 & 4\\0 & 1\end{pmatrix}$; $2A$ has rows $\begin{pmatrix}2 & 4\\0 & 2\end{pmatrix}$.

2. A row-start adjacency matrix C has rows $\begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{pmatrix}$. [5 marks]

(a) Calculate C^3. [3 marks]

The network is A→B→C→A.

Three steps return each start to itself.

Answer: The 3×3 identity matrix.

(b) Explain why all different vertices are reachable although $(C^3)_{AB}=0$. [2 marks]

A single power tests one exact length only.

Answer: Reachability allows other lengths: A reaches B in one step.

3. A two-vertex network has a loop at each vertex and an arrow A→B. Its adjacency rows are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. [5 marks]

(a) Find the number of two-step A-to-B walks and list them. [3 marks]

The (A, B) entry of L^2 is 2.

Answer: 2: A→A→B and A→B→B.

(b) How many three-step A-to-B walks are there? [2 marks]

The arrow A→B can occur at step 1,2 or 3, with loops at the other steps.

Answer: 3.

4. A dominance matrix has rows $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. A score is the row sum of $D + wD^2$. [5 marks]

(a) Express the three scores in terms of w. [3 marks]

Only A has an indirect two-step relation.

Answer: $(2 + w, 1, 0)$.

(b) For nonnegative w, can B overtake A under this rule? Explain. [2 marks]

The conclusion depends on the specified nonnegative weight.

Answer: No; $2 + w$ is at least 2, while B stays at 1.

5. A transition matrix is the 2×2 identity. [4 marks]

(a) Find all steady states totalling 100. [2 marks]

Every vector is unchanged.

Answer: $(a, 100 - a)$ for $0 \le a \le 100$.

(b) Does the model predict a unique long-run split independent of the start? [2 marks]

It retains the starting split forever.

Answer: No.

6. A transition matrix has rows $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and initial state $(80, 20)$. [5 marks]

(a) Find x_{10} and x_{11}. [2 marks]

Even powers restore and odd powers swap.

Answer: $x_{10} = (80, 20)$; $x_{11} = (20, 80)$.

(b) Find a steady state with the same total. Does this initial state approach it? [3 marks]

Balance requires equal entries; the oscillation persists.

Answer: Steady $(50, 50)$; the initial state alternates and does not converge.

7. Let T have rows $\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$, let S swap A and B, and start at $(80, 20)$. [6 marks]

(a) Apply T first, then S. [3 marks]

$T(80, 20) = (70, 30)$, then swap.

Answer: $(30, 70)$.

(b) Apply S first, then T, and compare. [3 marks]

Swap to $(20, 80)$, then T gives $(40, 60)$.

Answer: $(40, 60)$; the time order changes the forecast.

8. A closed 100-person group uses rows $\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$. The next state is $(68, 32)$. [6 marks]

(a) Recover the initial state. [4 marks]

Let initial A be a and B be $100 - a$.

Then $0.8a + 0.3(100 - a) = 68$, so $0.5a = 38$.

Answer: $(76, 24)$.

(b) Verify it by multiplication. [2 marks]

A: $60.8 + 7.2 = 68$; B: $15.2 + 16.8 = 32$.

Answer: $T(76, 24) = (68, 32)$.

9. An open model first applies rows $\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$, then adds 5 new members to A. Start at $(60, 40)$. [5 marks]

(a) Find the next state and total. [3 marks]

The closed transition preserves $(60, 40)$; add $(5, 0)$.

Answer: $(65, 40)$, total 105.

(b) Explain why writing only $x_1 = Tx_0$ would be inadequate. [2 marks]

Use $x_1 = Tx_0 + (5, 0)$.

Answer: It omits the external input and would incorrectly preserve total 100.

10. A report claims: “If every column sums to 1, the model is valid and every starting state approaches one accurate prediction.” [5 marks]

(a) Give two different mathematical objections. [4 marks]

Nonnegative probability bounds are also required.

Use identity for non-unique fixed states, or swapping for non-convergence.

Answer: Entries could be negative; identity or swap matrices need not yield a unique attracting limit.

(b) Give one empirical objection. [1 mark]

Internal consistency is not forecast validation.

Answer: Constant transition probabilities may not match future observed behaviour.