Matrix models - Practice test 2

10 questions | 45 marks | 70 minutes

Route networks, changing rules and evidence for a model. Network matrices use rows=start and columns=destination; transition matrices use columns=current and rows=next unless stated otherwise.

Equivalent correct methods earn the corresponding marks.

1. A sales matrix M has order 3×2 and a column price vector p has order 2×1. [4 marks]

(a) Which of Mp and pM is defined? Give its order. [2 marks]

For pM, inner dimensions 1 and 3 differ.

Answer: Only Mp, of order 3×1.

(b) A student instead multiplies each sale by one price chosen at random. Explain the missing idea. [2 marks]

The row-by-column dot product combines the labelled categories.

Answer: Each row must pair every product quantity with its matching price, then add.

2. For two locations A,B, rail matrix R has rows $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, and bus matrix S has rows $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Entries count direct services. [5 marks]

(a) Calculate RS. [3 marks]

Row 1: $0(1)+1(0)=0$ and $0(1)+1(1)=1$.

Row 2: $1(1)+1(0)=1$ and $1(1)+1(1)=2$.

Hence RS has rows $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$; row-start convention counts rail followed by bus.

Answer: $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

(b) How many rail-then-bus walks go A to B? Compare bus-then-rail. [2 marks]

Read $(RS)_{AB}=1$.

For SR, row 1 gives $1(0)+1(1)=1$ and $1(1)+1(1)=2$; row 2 gives $0(0)+1(1)=1$ and $0(1)+1(1)=1$.

Thus SR has rows $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, with two bus-then-rail A-to-B walks.

Answer: Rail then bus: 1; bus then rail: 2.

3. A weighted adjacency matrix W has rows $\begin{pmatrix}0&2&1\\0&0&3\\1&0&0\end{pmatrix}$, ordered A,B,C. Entries count different direct services. [4 marks]

(a) Find the number of two-service walks from A to C. [2 marks]

$0(1)+2(3)+1(0)=6$.

Answer: 6.

(b) Why is this not the number of distinct intermediate towns? [2 marks]

Two A-to-B choices combine with three B-to-C choices.

Answer: It counts choices of services; all six use B as the intermediate town.

4. A,B,C form an undirected path A-B-C. [5 marks]

(a) Calculate its squared adjacency matrix. [3 marks]

Row 1: $0(0)+1(1)+0(0)=1$; $0(1)+1(0)+0(1)=0$; $0(0)+1(1)+0(0)=1$.

Row 2: $1(0)+0(1)+1(0)=0$; $1(1)+0(0)+1(1)=2$; $1(0)+0(1)+1(0)=0$.

Row 3 equals row 1 because rows 1 and 3 of P are the same.

Thus P^2 has rows $\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$.

Answer: $\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$

(b) Count A-to-C walks using one, two or three edges. [2 marks]

The counts for lengths 1,2,3 are 0,1,0.

Answer: 1.

5. A four-vertex network has arrows A→B, B→C, C→A, C→D. D has no outgoing arrows. [3 marks]

(a) List the vertices reachable from B, excluding B itself. [2 marks]

B→C reaches C; extend to A or D.

Answer: A,C,D.

(b) Can D return to B? State what feature decides this. [1 mark]

Every nonzero-length walk from D is impossible.

Answer: No; D has no outgoing arrows.

6. Three teams form a dominance cycle A beats B, B beats C, C beats A. Score by row sums of $D + 2D^2$. [4 marks]

(a) Calculate each score. [2 marks]

Each has one direct win and one two-step relation: $1 + 2 = 3$.

Answer: 3 for each team.

(b) Suggest a fair way to report the result rather than invent a unique winner. [2 marks]

Changing the rule requires justification, not merely a desired winner.

Answer: Report a three-way tie under the stated rule and supply the cycle/results.

7. A hire firm uses transition rows $\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$ and starts with $(80, 40)$ bikes. [5 marks]

(a) Calculate tomorrow's expected state. [3 marks]

A: $56 + 8 = 64$; B: $24 + 32 = 56$.

Answer: $(64, 56)$.

(b) Interpret the matrix entry 0.3 and check the total. [2 marks]

The entry is row B,column A.

Answer: 30% move A to B; total is 120 before and after.

8. A population starts at $(100, 0)$. Period 1 uses T with rows $\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$; period 2 uses U with rows $\begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix}$. [6 marks]

(a) Write and calculate the correctly ordered forecast. [3 marks]

First $Tx_0 = (80, 20)$, then apply U.

Answer: $UTx_0 = (50, 50)$.

(b) Calculate the reversed-order forecast and interpret the difference. [3 marks]

First $Ux_0 = (60, 40)$, which is fixed under T.

Answer: $TUx_0 = (60, 40)$; it represents the reverse policy order.

9. A repair model has states Working,Faulty and transition rows $\begin{pmatrix} 1 & 0.2 \\ 0 & 0.8 \end{pmatrix}$. Start with $(0, 50)$. [5 marks]

(a) Find the state after 3 periods. [3 marks]

Faulty count is $50(0.8)^3 = 25.6$.

The remaining 24.4 are Working.

Answer: $(24.4, 25.6)$.

(b) Find the steady proportions and interpret the model's one-way assumption. [2 marks]

The balance equation forces the Faulty amount to zero.

Answer: $(1, 0)$; Working items never become Faulty in this model.

10. A closed model predicts $(60, 40)$, but later observations are $(58, 47)$. [4 marks]

(a) Compare the totals and identify a missing process. [2 marks]

Transfers alone preserve the total.

Answer: Prediction total 100, observed 105; entry/exit or counting/state changes must be checked.

(b) Would fitting a new matrix to this same observation independently validate it? Explain. [2 marks]

A fit and an independent test are different uses of evidence.

Answer: No; use separate later data for a forecast check.