Linear models and optimisation - Challenge and applications

10 questions | 49 marks | 75 minutes

A final paper requiring careful model criticism, integer reasoning and parameter comparisons.

Equivalent correct methods earn the corresponding marks.

1. A linear temperature model gives 18 degrees Celsius 2 hours after sunrise and 30 degrees 6 hours after sunrise. [4 marks]

(a) Find a rule $T = mt + b$, where t is hours after sunrise. [3 marks]

$m = \frac{30-18}{6-2} = 3$.

Use $18 = 3(2)+b$ to get $b = 12$.

Answer: $T = 3t + 12$.

(b) Explain why predicting the temperature at $t = 30$ needs caution. [1 mark]

A constant-rate rule need not hold over an entire day and night.

Answer: It extrapolates far outside the observed interval and may ignore daily cooling.

2. For a purchase model, $3x + 2y = 31$ and $x + y = 12$. Counts must be non-negative integers. [5 marks]

(a) Solve the equations. [3 marks]

Subtract twice the second equation from the first to get $x = 7$.

Then $y = 5$.

Answer: $(x, y) = (7, 5)$.

(b) If the total cost were 30 rather than 31, what would the solution be? [2 marks]

Use $3x + 2y = 30$ and $x + y = 12$; subtract to obtain $x = 6$.

Answer: $(6, 6)$.

3. A stall sells an item for AUD 17 and incurs AUD 92 fixed cost plus AUD 6 per item. [5 marks]

(a) Find its profit rule. [2 marks]

Revenue $17n$ minus cost $92+6n$ gives $11n-92$.

Answer: $P=11n-92$.

(b) Find the smallest whole-number sales quantity giving at least AUD 100 profit. [3 marks]

$11n-92\geq 100$ gives n at least $\frac{192}{11}$, about 17.45.

Round up and verify $P(18)=106$.

Answer: 18 items.

4. Solve $5-2(3x-1)>19-x$. [4 marks]

(a) Give the solution set. [3 marks]

$7-6x>19-x$ gives $-5x>12$; reverse when dividing by -5.

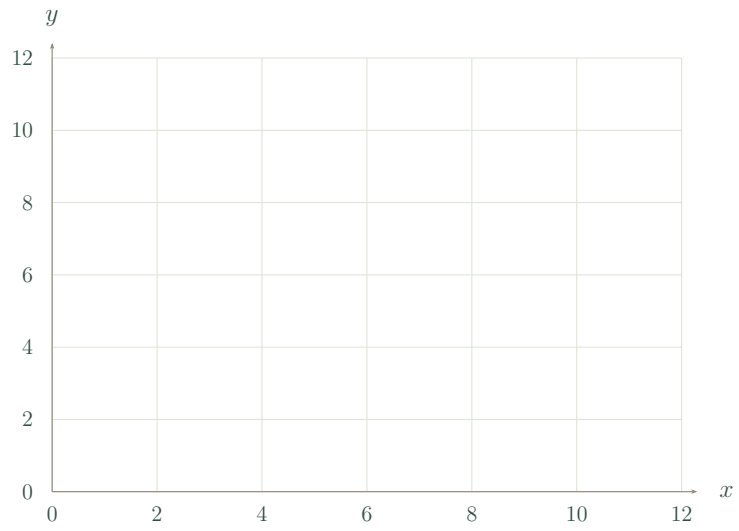
Answer: $x<-\frac{12}{5}$.

(b) Find the greatest integer solution. [1 mark]

-2 is too large; -3 is the greatest integer below -2.4.

Answer: -3.

5. A feasible region satisfies $x \geq 2$, $y \geq 1$, $x + y \leq 10$ and $2x + y \leq 14$. [7 marks]



(a) List all its vertices. [4 marks]

Intersect the lower bounds with each active upper boundary.

Discard intersections that fail another inequality.

Answer: $(2, 1), (\frac{13}{2}, 1), (4, 6), (2, 8)$.

(b) Maximise $P = 7x + 4y$. [3 marks]

Vertex values are 18,49.5,52,46.

Answer: Maximum 52 at$(4, 6)$.

6. For the region in Question 5, decisions must instead be whole numbers. [3 marks]

(a) Does the continuous optimum already satisfy integrality? [1 mark]

Both 4 and 6 are integers.

Answer: Yes.

(b) Explain why it is also an integer optimum without enumerating every integer pair. [2 marks]

This argument works because the continuous maximiser is itself an allowed integer point.

Answer: Integer feasible points are a subset of the continuous region; none can exceed its continuous maximum 52, which$(4, 6)$ attains.

7. Minimise $C = 5x + 4y$ with $x, y \geq 0$, $x + y \geq 9$ and $2x + y \geq 13$. [6 marks]

(a) Find the lower-bound vertices. [3 marks]

Intersect the two sloping boundaries to get$(4, 5)$; use the tighter axis bounds.

Answer: $(9, 0), (4, 5), (0, 13)$.

(b) Find the minimum and interpret unused excess in each minimum requirement. [3 marks]

Costs are 45,40,52.

The sums are 9 and 13 at the optimum.

Answer: Minimum 40 at$(4, 5)$; both requirements are met exactly.

8. A bounded integer search gives these best feasible y-values for $x = 0, 1, 2, 3, 4$:5,4,4,2,0. Let $P = 4x + 3y$. [5 marks]

(a) Find the maximum over the listed candidates. [3 marks]

The values are 15,16,20,18,16.

Answer: 20 at$(2, 4)$.

(b) What fact must be established before this table proves the global integer optimum? [2 marks]

A table of only convenient candidates is not an exhaustive proof.

Answer: That $x = 0$ through 4 is the full permitted integer range and the selected y gives the greatest objective for each x.

9. A feasible region is defined only by $x, y \geq 0$. A report gives a finite maximum for $P = 2x + y$. [3 marks]

(a) Explain why the report cannot be correct for this stated region. [2 marks]

For example $(t, 0)$ gives $2t$ without limit.

Answer: Both variables can increase and the objective is unbounded above.

(b) State one kind of real information that could justify a finite model. [1 mark]

The bound must come from the context, not a desired numerical answer.

Answer: A supported capacity, resource or demand upper bound.

10. A feasible polygon has vertices $(0, 0), (8, 0), (4, 6), (0, 8)$. Profit is $P = px + 4y$, where p is positive. [7 marks]

(a) Find the p-values for which$(4, 6)$ is an optimum. [4 marks]

Compare $4p + 24$ with $8p$: p at most 6.

Compare $4p + 24$ with 32: p at least 2; it also exceeds the origin for positive p.

Answer: $2 \leq p \leq 6$.

(b) When is$(4, 6)$ the unique optimum, and what happens at the endpoints? [3 marks]

Equality in a vertex comparison makes the objective constant along the connecting feasible edge.

Answer: Unique for $2 < p < 6$. At $p = 2$ the edge to$(0, 8)$ ties; at $p = 6$ the edge to$(8, 0)$ ties.