Matrices and networks - Applications and reasoning test

10 questions | 60 marks | 75 minutes

An original mixed chapter paper. Show calculations, state units and justify network decisions using the supplied information.

Equivalent correct methods earn the corresponding marks.

1. A stock matrix is $S=\begin{pmatrix} x & 3 \\ 2 & y \end{pmatrix}$. Its row totals are 10 and 8 items. [6 marks]

(a) Find x and y, explaining which totals you used. [3 marks]

Row 1 gives $x+3=10$.

Row 2 gives $2+y=8$.

Answer: $x=7,\ y=6$.

(b) Unit values for its two columns are 4 and 5 dollars. Calculate each row's stock value. [3 marks]

Use price column $(4,5)^T$.

$7(4)+3(5)=43$ and $2(4)+6(5)=38$.

Answer: 43 and 38 dollars.

2. Two orders of two product types cost 31 and 37 dollars. The quantity rows are $(2,3)$ and $(4,1)$. Let the unit prices be p and q. [6 marks]

(a) Form the matrix equation and solve for the prices. [3 marks]

Equations are $2p+3q=31$ and $4p+q=37$.

Substitute $q=37-4p$: $-10p=-80$.

Then $p=8,\ q=5$.

Answer: $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} 31 \\ 37 \end{pmatrix};\ p=8,\ q=5$.

(b) Check the prices in both orders and explain why the result should be interpreted as a model assumption. [3 marks]

$2(8)+3(5)=31;\ 4(8)+1(5)=37$.

Bulk discounts or delivery fees could invalidate a fixed-price model.

Answer: Both costs are reproduced; it assumes a fixed unit price and no other fees.

3. Rows of $Q=\begin{pmatrix}2 & 1 & 3\\ 0 & 4 & 2\end{pmatrix}$ describe two bundles. Product prices at suppliers 1 and 2 are columns of $P=\begin{pmatrix}2 & 5\\ 3 & 4\\ 1 & 2\end{pmatrix}$ dollars. [6 marks]

(a) Calculate QP and interpret all four entries. [3 marks]

Row 1 costs 10 at supplier 1 and 20 at supplier 2.

Row 2 costs 14 at supplier 1 and 20 at supplier 2.

Answer: $\begin{pmatrix}10 & 20\\ 14 & 20\end{pmatrix}$ dollars.

(b) Supplier 1 charges 5 dollars delivery per bundle, supplier 2 none. Recommend a supplier for each bundle. [3 marks]

Add 5 to each supplier 1 product subtotal.

Compare full costs, not subtotals alone.

Answer: Supplier 1 for both:15 versus 20 and 19 versus 20 dollars.

4. A theatre’s ticket-quantity rows are matinee/evening and columns adult/student. $Q=\begin{pmatrix}12 & 8\\ 20 & 15\end{pmatrix}$. Adult tickets cost 18 dollars; student tickets 12 dollars. [6 marks]

(a) Use a matrix product to find revenue for each performance. [3 marks]

$12(18)+8(12)=312.$

$20(18)+15(12)=540.$

Answer: 312 and 540 dollars.

(b) A sponsor pays 2 dollars for each student ticket. Find total revenue including this contribution, and state an assumption. [3 marks]

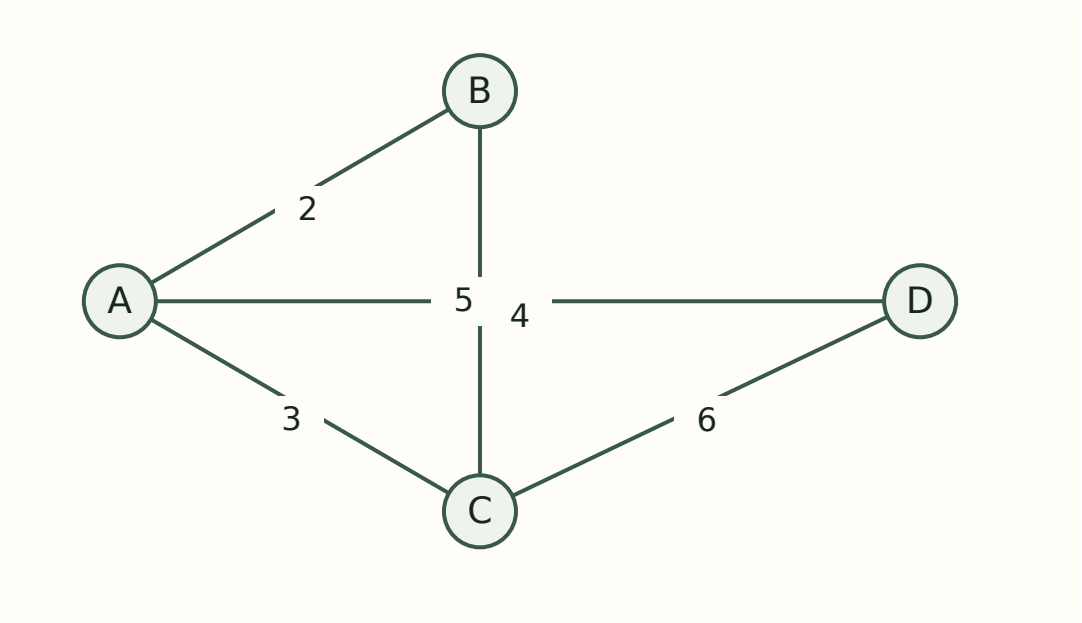
Ticket revenue is 852 dollars.

Student tickets total 23; sponsor adds 46.

$852+46=898.$

Answer: 898 dollars, assuming all recorded tickets are paid and contributions apply to each student ticket.

5. The undirected weights are cable-installation costs. [6 marks]



(a) Find a minimum spanning tree and identify its bridges. [3 marks]

These three cheapest connections link every vertex without a cycle.

Removing any tree edge separates a site.

Answer: A-B 2, A-C 3, A-D 4; total 9; every selected edge is a bridge.

(b) A backup B-D edge is added. Which selected original edge remains a bridge, and why? [3 marks]

The new cycle is A-B-D-A.

C still has only its selected A-C connection.

Answer: A-C remains a bridge.

6. The directed transport links are listed. Time is minutes; capacity is passengers/min. [6 marks]

All available directed transport links

Link	Time	Capacity
S→A	2	4
A→T	3	4
S→B	4	7
B→T	4	7
A→B	1	2

(a) Find a fastest S-to-T path and its route bottleneck. [3 marks]

That time is $2+3=5$.

The smaller capacity on the route is 4.

Answer: S→A→T takes 5 min; route capacity 4 passengers/min.

(b) Find maximum total throughput using both main branches. Explain why it differs from the fastest route’s capacity. [3 marks]

Send 4 through A and 7 through B.

The source cut is $4+7=11$, proving the maximum.

Fastest individual travel and highest total throughput are different objectives.

Answer: 11 passengers/min.

7. Rows/columns A, B, C in $N=\begin{pmatrix}0 & 2 & 1\\0 & 0 & 3\\0 & 0 & 0\end{pmatrix}$ count distinct scheduled services between stops.

[6 marks]

(a) Count exactly two-service journeys A-to-C and explain the calculation. [3 marks]

Choose one of 2 A→B services, then one of 3 B→C services.

$2(3)=6$, also $(N^2)_{AC}$.

Answer: 6.

(b) Count all A-to-C service choices with no repeated stop, allowing one or two services. Why would the same matrix calculation not count journeys if its entries were travel times? [3 marks]

The one-edge journey adds 1 to the six two-edge choices.

Matrix-power walk counts require adjacency multiplicities, not time weights.

Answer: 7 choices; direct 1 plus via-B 6. Travel-time weights are not counts of service choices.

8. Directed edges are S→A 3, S→B 4, A→C 5, B→C 1, A→T 6, C→T 4 and B→T 7. [6 marks]

(a) Find a longest S-to-T path in this acyclic graph and justify your answer. [3 marks]

The four path totals are 9, 12, 11 and 9.

The maximum is 12.

Answer: S→A→C→T;12.

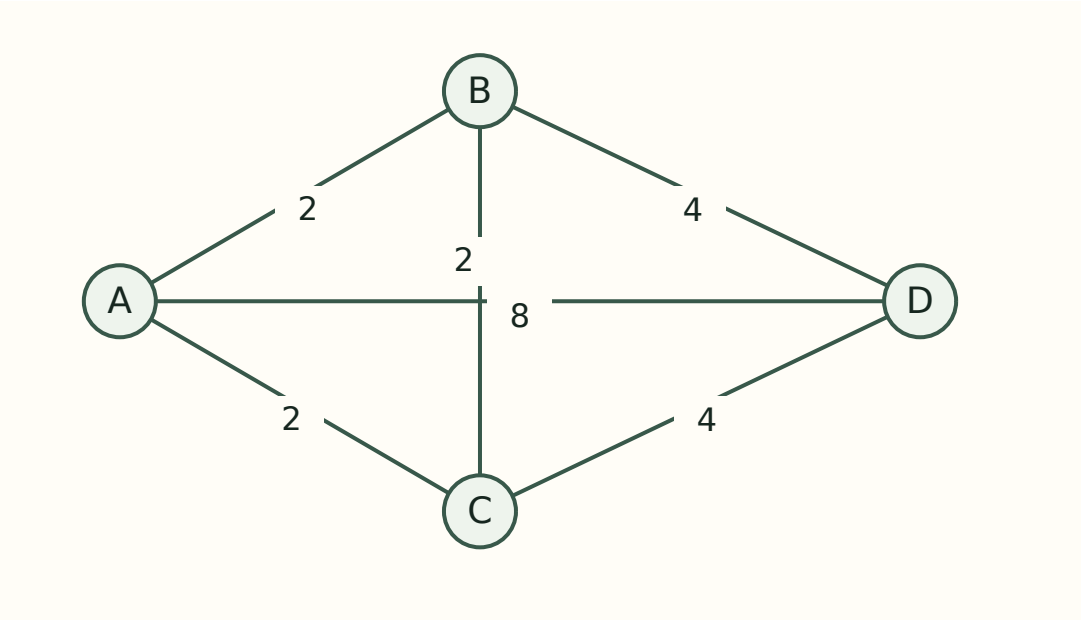
(b) A positive-weight edge C→A is added. Explain why a longest walk, with repetition permitted, now has no finite maximum. [3 marks]

The source reaches A and the cycle can then be repeated.

Each repetition adds positive weight before finishing at T.

Answer: A→C→A is a repeatable positive cycle with access to T.

9. Use the undirected weighted network shown. [6 marks]



(a) Find one minimum spanning tree and show why its total cannot be less. [3 marks]

Connecting D requires an edge of at least 4.

Two further edges are needed to connect A, B, C, each costing at least 2.

The lower bound $4+2+2=8$ is achieved.

Answer: For example A-B 2, B-C 2, C-D 4; total 8.

(b) How many different minimum trees are possible? Explain systematically. [3 marks]

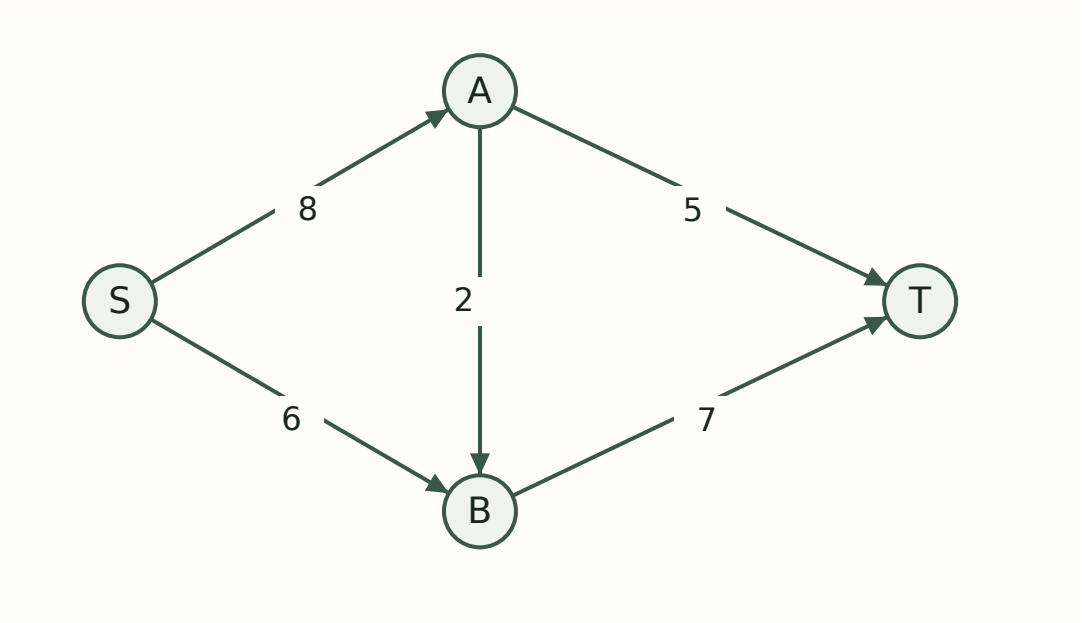
Choose any 2 of the 3 equal-cost triangle edges:3 choices.

Join D using B-D or C-D:2 choices.

$3(2)=6$ trees.

Answer: 6.

10. Pipe capacities are units/hour. You may increase exactly one capacity by 2. [6 marks]



(a) Find and certify the original maximum flow. [3 marks]

Use SA 7, SB 5, AB 2, AT 5, BT 7.

Conservation holds and the sink cut has capacity $5+7=12$.

Answer: 12.

(b) Choose between widening S→A or A→T by 2. Prove which gives the larger new maximum. [3 marks]

Changing S→A leaves the sink cut 12.

With A→T 7, use SA 8, SB 6, AB 1, AT 7, BT 7 to achieve 14.

The source cut is $8+6=14$, proving optimality.

Answer: Widen A→T; new maximum 14. Widening S→A alone leaves maximum 12.